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A THEORY OF CONVERGENCE

E. J. McSHANE

The literature already contains several theories of limits which have great generality (**10**; **12**; **1**; **5**; **6**; **13**; **3**; **14**; **4**, p. 34). Nevertheless, the intrinsic importance and frequent use of the concept may justify the publication of another variant, provided that it has advantages in ease of application without sacrifice of generality. The theory of convergence studied in this note includes the other theories and their applications in a smooth way, without artifice.

1. Definition. If f is a function on a topological space D to a topological space R, and $a \in D$ and $b \in R$, the standard definition of

$$\lim_{x \to a} f(x) = b$$

can be phrased thus. Let $\mathfrak{N}$ be the family of all neighborhoods of a and $\mathfrak{Y}$ the family of all neighborhoods of b; to each $Y \in \mathfrak{Y}$ corresponds $N \in \mathfrak{N}$ such that if $x \in N$ then $f(x) \in Y$. Similarly, if the domain D of f is directed by $\succ$, so that $f, \succ$ is a net of points of S, the convergence of this net to $b \in S$ can be thus expressed. Each $n \in D$ defines a "final section" $D[\succ n]$ of D, where $D[\succ n]$ means the set $\{m \mid m \in D \,\&\, m \succ n\}$. Let $\mathfrak{N}$ consist of all final sections of D, and let $\mathfrak{Y}$ consist of all neighborhoods of b. Then

$$\lim_{n,\, \succ} f(n) = b$$

if and only if to each $Y \in \mathfrak{Y}$ corresponds a final section $D[\succ n] \in \mathfrak{N}$ such that if $m \in D[\succ n]$ then $f(m) \in \mathfrak{Y}$. These definitions and others too have the common form that we have given a function f, a family $\mathfrak{N}$ of subsets of the domain D_f of f, and another family $\mathfrak{Y}$ of sets, and the convergence of the function takes place if to each set $Y \in \mathfrak{Y}$ corresponds a set $N \in \mathfrak{N}$ such that if $x \in N$ then $f(x) \in Y$. Under these circumstances we shall say that "f converges over $\mathfrak{N}$ into $\mathfrak{Y}$," or "$f(x)$ converges into $\mathfrak{Y}$ as x converges into $\mathfrak{N}$."

Some theorems can be proved with no further assumptions. However, we prefer to devote most of our attention to families $\mathfrak{N}$ having properties which will enable us to prove a few of the most elementary theorems concerning limits of real-valued functions. Specifically, let $\mathfrak{N}$ be a family of subsets of some set D. We wish to find the weakest hypotheses on $\mathfrak{N}$ that will enable us to prove the following three statements.

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(i) *Some real-valued function on D converges over $\mathfrak{N}$ to some real number.*
(ii) *Some real-valued function on D does not converge over $\mathfrak{N}$ to every real number.*
(iii) *If f_1 and f_2 are real-valued functions on D, both of which converge over $\mathfrak{N}$ to 1, then $f_1 + f_2$ converges over $\mathfrak{N}$ to 2.*

It is easily seen that (i) is satisfied if and only if $\mathfrak{N}$ is non-empty, and (ii) is satisfied if and only if each $N \in \mathfrak{N}$ is non-empty. To investigate the effect of (iii), we let N_1 and N_2 be any members of $\mathfrak{N}$, and let f_1 and f_2 be their respective characteristic functions; then $f_1(x)$ and $f_2(x)$ converge over $\mathfrak{N}$ to 1. If (iii) holds, their sum converges to 2; therefore there exists N_3 in $\mathfrak{N}$ such that if $x \in N_3$, then

$$|f_1(x) + f_2(x) - 2| < 1.$$

But this last statement holds on the intersection $N_1 \cap N_2$ and nowhere else. Hence $N_3 \subset N_1 \cap N_2$. Conversely, if $N_1 \cap N_2$ contains a member of $\mathfrak{N}$, $f_1 + f_2$ converges over $\mathfrak{N}$ to 2, and (iii) holds.

In the terminology of H. Cartan [5], a family $\mathfrak{A}$ of sets is a *filter-base* if and only if $\mathfrak{A}$ is a non-empty family of non-empty sets such that if A and B are in $\mathfrak{A}$, there exists a set C of $\mathfrak{A}$ which is contained in A and in B. Therefore the results of the preceding paragraph may be summarized thus: *in order that* (i), (ii), *and* (iii) *be satisfied, it is necessary and sufficient that $\mathfrak{N}$ be a filter-base.*

If $\mathfrak{N}$ is a filter-base in the domain D_f of a function f, the range of f (structureless in itself) is organized or systematized by classifying its points into the images of the various N of $\mathfrak{N}$. Since *σύνταξις* means "putting together in order, arranging, . . . ; system, arrangement, organization, . . ." (Liddell and Scott, *Greek-English Lexicon*), we adopt the following definition.

(1) DEFINITION. *A* syntax *is a system $\langle f; \mathfrak{N} \rangle$ in which f is a function and $\mathfrak{N}$ is a filter-base in the domain D_f of f. When $\langle f; \mathfrak{N} \rangle$ is a syntax and the values of f are in a set S, $\langle f; \mathfrak{N} \rangle$ is a* syntax of points of S.

Thus, for example, if $\mathfrak{N}$ consists of all neighborhoods of the number π, the syntax $\langle \sin; \mathfrak{N} \rangle$ is an appropriate tool in the study of the behavior of the sine-function near π. To study its behavior near some other point x_0 we would need to replace $\mathfrak{N}$ by the family of neighborhoods of x_0, producing a different syntax.

Not all our theorems require the full strength of definition (1). Definitions and theorems (2) to (12), except for (8) and with (12 ii) removed, are still valid if in (1) we replace the words "filter-base in" by "family of subsets of." Moreover, (8) and (12) in full hold if $\mathfrak{N}$ is assumed to be a family of subsets of D_f such that each pair of sets in N has a non-empty intersection. However, there does not appear to be enough advantage in this extra generality to warrant giving it any further attention.

(2) DEFINITION. *If $\langle f; \mathfrak{N} \rangle$ is a syntax and $\mathfrak{Y}$ is a family of sets, $\langle f; \mathfrak{N} \rangle$* converges into *$\mathfrak{Y}$ if and only if to each set $Y \in \mathfrak{Y}$ corresponds a set $N \in \mathfrak{N}$ such that if $x \in N$, then $f(x) \in Y$.*

Alternative wordings for "$\langle f; \mathfrak{R}\rangle$ converges into $\mathfrak{Y}$" are "f converges over $\mathfrak{R}$ into $\mathfrak{Y}$" and "$f(x)$ converges into $\mathfrak{Y}$ as x converges into $\mathfrak{R}$."

In many applications there is a unique point common to the sets Y in $\mathfrak{Y}$. This point, determined by $\mathfrak{Y}$, we could designate by $[\mathfrak{Y}]$. Even when the family $\mathfrak{Y}$ does not determine such a unique point, it still may be regarded as defining an ideal point $[\mathfrak{Y}]$ in some space of sets. A similar statement holds for $\mathfrak{R}$. Accordingly, the sentence "$\langle f; \mathfrak{R}\rangle$ converges into $\mathfrak{Y}$" can be symbolized in any one of the following three ways:

$$f(x) \to [\mathfrak{Y}] \text{ as } x \to [\mathfrak{R}], \qquad \langle f; \mathfrak{R}\rangle \to [\mathfrak{Y}],$$

$$\lim_{x \to [\mathfrak{R}]} f(x) = [\mathfrak{Y}].$$

(For this suggestion, I owe thanks to the referee.)

Remark 1. If we wish to discuss multiple-valued functions f, we could correspondingly change Definition 2, replacing the final "$f(x) \in Y$" by "all values of $f(x)$ are in Y." This would furnish a generalization of a mode of convergence of multiple-valued functions which has occasionally been used.

Remark 2. We can conveniently adapt Halmos's terminology, saying that a statement concerning $f(x)$ is "eventually" true if it is true of $f(x)$ for all x in some set $N \in \mathfrak{R}$. Thus (2) becomes "$\langle f; \mathfrak{R}\rangle \to [\mathfrak{Y}]$ if and only if for each $Y \in \mathfrak{Y}$, $f(x)$ is eventually in Y."

Remark 3. If a syntax $\langle f; \mathfrak{R}\rangle$ converges into $\mathfrak{Y}$, we may assume without loss of generality that $\mathfrak{Y}$ is a filter-base, since the set $\mathfrak{Y}'$ of all intersections of finite subfamilies of $\mathfrak{Y}$ is then a filter-base and $\langle f; \mathfrak{R}\rangle \to [\mathfrak{Y}']$.

It is easy to see that the usual theorems on limits of sums, products, etc., of real functions can be established in the customary manner. If S is a set partially ordered by $\succ$, let $D[\succ a]$ denote the set $\{s \mid s \in S \,\&\, s \succ a\}$. Such sets will be called "final sections" of S. If we wish to avoid adjoining ∞ to the reals, we can notice that "$f(x) \to \infty$" is the same as "$f(x)$ converges into the final sections of the real number system." The final sections of S form a filter-base if and only if S is directed by $\succ$, so convergence of nets is a special case of convergence of syntaxes. The convergence defined by Bennett **(1)** is similarly covered. If f is on a topological space D_f to the extended real numbers, it is lower semi-continuous at a if and only if as $x \to a$, $f(x)$ converges into the family $\mathfrak{Y}$ of open sets containing $\{y \mid y \succcurlyeq f(a)\}$.

2. Subsyntaxes. In order to define the concept of subsyntax it is convenient first to define the relation "finer than."

(3) DEFINITION. *Let $\mathfrak{M}$, $\mathfrak{R}$ be families of sets. Then $\mathfrak{M}$ is* finer than $\mathfrak{R}$ *if and only if each $N \in \mathfrak{R}$ contains a set $M \in \mathfrak{M}$.*

(4) DEFINITION. *Let f be a function, and let $\mathfrak{M}$, $\mathfrak{N}$ be filter-bases in the domain of f. Then $\langle f; \mathfrak{M}\rangle$ is a* subsyntax *of $\langle f; \mathfrak{N}\rangle$ if and only if $\mathfrak{M}$ is finer than $\mathfrak{N}$.*

(5) COROLLARY. *If $\langle f; \mathfrak{N}''\rangle$ is a subsyntax of $\langle f; \mathfrak{N}'\rangle$, and the latter is a subsyntax of $\langle f; \mathfrak{N}\rangle$, then $\langle f; \mathfrak{N}''\rangle$ is a subsyntax of $\langle f; \mathfrak{N}\rangle$.*

(6) COROLLARY. *If $\langle f; \mathfrak{M}\rangle$ is a subsyntax of $\langle f; \mathfrak{N}\rangle$, and $\mathfrak{Y}$ is a family of sets, and f converges over $\mathfrak{N}$ into $\mathfrak{Y}$, then f also converges over $\mathfrak{M}$ into $\mathfrak{Y}$.*

As a special case, suppose that $\langle f; \mathfrak{N}\rangle$ is a syntax and A is a subset of D_f such that $A \cap N$ is non-empty for every $N \in \mathfrak{N}$. We define $\mathfrak{M}$ to be the family of all such intersections $A \cap N$; then $\mathfrak{M}$ is easily seen to be a filter-base which is finer than $\mathfrak{N}$. Hence $\langle f; \mathfrak{M}\rangle$ is a subsyntax of $\langle f; \mathfrak{N}\rangle$. When D_f is the set of positive integers, and A is an infinite subset $\{n_1, n_2 \ldots\}$ of D_f, the subsyntax $\langle f; \mathfrak{M}\rangle$ does not differ in any essential way from the traditional subsequence

$$\{f(n_1), f(n_2), \ldots\}.$$

A similar statement holds if D_f is directed by a relation $\succ$ and A is a cofinal subset. But this method of constructing subsyntaxes is by no means the only one possible, as we shall see in several later theorems (e.g., Theorems (11) and (14)).

3. Cluster-points. Our next definition generalizes the concept of cluster-point of a sequence.

(7) DEFINITION. *Let $\langle f; \mathfrak{N}\rangle$ be a syntax and $\mathfrak{Y}$ a family of sets. Then* $\langle f; \mathfrak{N}\rangle$ clusters at $[\mathfrak{Y}]$ *if and only if for each $N \in \mathfrak{N}$ and each $Y \in \mathfrak{Y}$ there exists $x \in N$ such that $f(x) \in Y$.*

In particular, when $\mathfrak{Y}$ is the family of neighborhoods of a point p in a topological space and $\langle f; \mathfrak{N}\rangle$ clusters at $[\mathfrak{Y}]$ we say that p is a cluster-point of $\langle f; \mathfrak{N}\rangle$.

(8) COROLLARY. *If $\mathfrak{Y}$ is a family of sets and $\langle f; \mathfrak{N}\rangle$ a syntax which converges into $\mathfrak{Y}$, then $\langle f; \mathfrak{N}\rangle$ clusters at $[\mathfrak{Y}]$.*

For if $N \in \mathfrak{N}$ and $Y \in \mathfrak{Y}$, there exists $N' \in \mathfrak{N}$ such that $f(N') \subset Y$; N and N' have a point x in common, for which $x \in N$ and $f(x) \in Y$.

(9) COROLLARY. If $\langle f; \mathfrak{M}\rangle$ *is a subsyntax of* $\langle f; \mathfrak{N}\rangle$, *and* $\langle f; \mathfrak{M}\rangle$ *clusters at* $[\mathfrak{Y}]$, *so does* $\langle f; \mathfrak{N}\rangle$.

For let $Y \in \mathfrak{Y}$ and $N \in \mathfrak{N}$. Then N contains a set M of the family $\mathfrak{M}$, and there exists $x \in M$ such that $f(x) \in Y$.

The following rather obvious statement is needed in several proofs.

(10) LEMMA. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be filter-bases such that for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ the intersection $A \cap B$ is non-empty. Let $\mathfrak{C}$ consist of all sets of the form $A \cap B$, $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Then $\mathfrak{C}$, $\mathfrak{A} \cup \mathfrak{C}$ and $\mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C}$ are filter-bases.*

(11) THEOREM. *If $\langle f; \mathfrak{N}\rangle$ is a syntax and $\mathfrak{Y}$ is a filter-base and $\langle f; \mathfrak{N}\rangle$ clusters at $[\mathfrak{Y}]$, there exists a subsyntax $\langle f; \mathfrak{M}\rangle$ of $\langle f; \mathfrak{N}\rangle$ which converges into $\mathfrak{Y}$.*

Proof. Let $\mathfrak{M}$ consist of all intersections $N \cap f^{-1}(Y)$ with $N \in \mathfrak{N}$ and $Y \in \mathfrak{Y}$. By (10) this is a filter-base, and it is obviously finer than $\mathfrak{N}$; and for each Y in $\mathfrak{Y}$, for all x in $N \cap f^{-1}(Y)$ (N an arbitrary fixed member of $\mathfrak{N}$) we find $f(x)$ in Y.

Remark 1. Under the hypotheses of (11) there exists a net $(g(\alpha) \mid \alpha \in A)$, $\succ$ of points of D_f such that g, $\succ$ converges into $\mathfrak{N}$ and $(f(g(\alpha)) \mid \alpha \in A)$, $\succ$ converges into $\mathfrak{Y}$. For we can choose A to be the family $\mathfrak{M}$ of intersections and for each $\alpha \in A$ choose $g(\alpha) \in \alpha$.

Remark 2. Still under the hypotheses of (11), if $\mathfrak{N}$ and $\mathfrak{Y}$ have countable subsets $\{N_i \mid i = 1, 2, \dots\}$ and $\{Y_i \mid i = 1, 2, \dots\}$ which are finer than $\mathfrak{N}$ and $\mathfrak{Y}$ respectively, there is a sequence $(g(m) \mid m = 1, 2, \dots)$ of points D_f such that as $m \to \infty$, $g(m)$ converges into $\mathfrak{N}$ and $f(g(m))$ into $\mathfrak{Y}$. For we can choose

$$g(m) \in X_m \cap f^{-1}(Y_m), \qquad m = 1, 2, \dots.$$

4. Uniqueness. Let us first dispose of the somewhat trivial question of the convergence of syntaxes $\langle f; \mathfrak{N}\rangle$ such that f has a constant value p on some set N of the family $\mathfrak{N}$. It is evident that such a syntax converges into the sets of a family $\mathfrak{Y}$ if and only if p is in every set Y of $\mathfrak{Y}$. In particular, if $\mathfrak{Y}$ is the family of all neighborhoods of a point q in a topological space S, this syntax $\langle f; \mathfrak{N}\rangle$ converges to q if and only if p is in every neighborhood of q. Thus if p is in a topological space S, and $\langle f; \mathfrak{N}\rangle$ is a syntax, and f is constantly equal to p on some set N of $\mathfrak{N}$, then $\langle f; \mathfrak{N}\rangle$ converges to p; and it converges to p alone if S is a T_1-space.

In the next theorem and several others we need to make use of the identity-function, which at each x in the universe of discourse has the functional value x. For this function we shall use the name id; thus id $x = x$ for all x.

(12) THEOREM. *Let $\mathfrak{Y}, \mathfrak{Y}'$ be filter-bases. The following conditions are equivalent.*

(i) *For every $Y \in \mathfrak{Y}$ and every $Y' \in \mathfrak{Y}$, $Y \cap Y'$ is non-empty.*

(ii) *There exists a syntax $\langle f; \mathfrak{N}\rangle$ which converges into $\mathfrak{Y}$ and into $\mathfrak{Y}'$.*

(iii) *There exists a syntax $\langle f; \mathfrak{N}\rangle$ which converges into $\mathfrak{Y}$ and clusters at $[\mathfrak{Y}']$.*

Proof. By (8), (ii) implies (iii). Let (iii) hold, and let $Y \in \mathfrak{Y}$ and $Y \in \mathfrak{Y}'$. For some $N \in \mathfrak{N}$, $f(N) \subset Y$, and there exists $x \in N$ such that $f(x) \in Y'$, so $f(x) \in Y \cap Y'$, and (i) holds. If (i) holds, let $\mathfrak{N}$ be the set of all intersections $Y \cap Y'$ with $Y \in \mathfrak{Y}$ and $Y' \in \mathfrak{Y}'$. By (10), this is a filter-base; it is finer than $\mathfrak{Y}$ and $\mathfrak{Y}'$, so $\langle \mathrm{id}; \mathfrak{N}\rangle$ is a subsyntax of $\langle \mathrm{id}; \mathfrak{Y}\rangle$ and $\langle \mathrm{id}; \mathfrak{Y}'\rangle$. Since it is evident that these syntaxes converge into $\mathfrak{Y}$ and $\mathfrak{Y}'$ respectively, by (6) $\langle \mathrm{id}; \mathfrak{N}\rangle$ converges into $\mathfrak{Y}$ and into $\mathfrak{Y}'$. Hence (i) implies (ii).

In particular, if $\mathfrak{Y}$ and $\mathfrak{Y}'$ are the families of neighborhoods of points p, p' in a topological space, there exists a syntax $\langle f; \mathfrak{N}\rangle$ converging to both p and p' (or converging to one and having the other as cluster-point) if and only if

every neighborhood of p meets every neighborhood of p'. This implies $p = p'$ if and only if the space is a Hausdorff space.

5. Decided Syntaxes. We adapt to syntaxes an expressive phrase of Tukey's **(14)**. If $\langle f; \mathfrak{N} \rangle$ is a syntax and Y a set, the syntax *decides for* Y if $f(x)$ is eventually in Y, and *decides against* Y if $f(x)$ is eventually not in Y; in either case, $\langle f; \mathfrak{N} \rangle$ *decides about* Y. A syntax will be called a *decided syntax* if it decides about every set Y.

(13) DEFINITION. *A syntax* $\langle f; \mathfrak{N} \rangle$ *is* decided *if to every set* Y *corresponds* $N \in \mathfrak{N}$ *having one of the following properties*:

(i) *for all* $x \in N$, $f(x) \in Y$; *or*

(ii) *for all* $x \in N$, $f(x) \notin Y$.

The "decided syntaxes" are the analogues of the "ultra-filters" of Cartan **(4)** and the "universal nets" of Kelley **(7)**.

Remark 1. Clearly (13) is unaltered if we consider only sets Y contained in the range of f.

Remark 2. The syntax $\langle f; \mathfrak{N} \rangle$ is decided if and only if the images $f(N)$, $N \in \mathfrak{N}$ form an ultra-filter.

(14) THEOREM. *Every syntax has a decided subsyntax.*

Let $\langle f; \mathfrak{N} \rangle$ be a syntax. The collection Φ of all filter-bases in D_f which contain $\mathfrak{N}$ is partially ordered by $\supset$. If Φ_0 is a linearly ordered subset of Φ, the union $\mathfrak{U}$ of all the members of Φ_0 is itself a member of Φ. By the Hausdorff maximal principle (Zorn's lemma) there exists a maximal member $\mathfrak{M}$ of Φ. Then $\langle f; \mathfrak{M} \rangle$ is a syntax, and since each $N \in \mathfrak{N}$ contains (in fact, is) a member of M, $\langle f; \mathfrak{M} \rangle$ is a subsyntax of $\langle f; \mathfrak{N} \rangle$. To prove it a decided syntax let Y be any set. If there is no $M \in \mathfrak{M}$ such that $f(M) \cap Y$ is empty, let $\mathfrak{B}$ consist of $f^{-1}(Y)$ alone. This is a filter-base, and for all $M \in \mathfrak{M}$ and all $B \in \mathfrak{B}$ we know that $M \cap B$ is not empty. The set $\mathfrak{C}$ of all such intersections is a filter-base by (10), and so is $\mathfrak{M} \cup \mathfrak{B} \cup \mathfrak{C}$. This contains the maximal filter-base $\mathfrak{M}$, so it is identical with $\mathfrak{M}$, and $\mathfrak{B} \subset \mathfrak{M}$; that is, $f^{-1}(Y)$ is a member M' of $\mathfrak{M}$. Hence $f(M') \subset Y$, which completes the proof.

(15) THEOREM. *If* $\mathfrak{Y}$ *is a family of sets, and a decided syntax* $\langle f; \mathfrak{N} \rangle$ *clusters at* $[\mathfrak{Y}]$, *it converges into* $\mathfrak{Y}$.

For each $Y \in \mathfrak{Y}$, $f(N) \cap Y$ is non-empty for all $N \in \mathfrak{N}$, so by (13) there exists $N \in \mathfrak{N}$ such $f(N) \subset Y$.

6. Characterizations of Compactness. Syntaxes serve as well as nets or filters in characterizing compactness.

(16) THEOREM. *Let* S *be a topological space. Then the following statements are equivalent.*

(i) S *is compact.*

(ii) *Every syntax of points of* S *has a cluster point in* S.

(iii) *For every function f on S to S and every filter-base $\mathfrak{N}$ in S, the syntax $\langle f; \mathfrak{N}\rangle$ has a cluster point in S.*

(iv) *For every filter-base $\mathfrak{N}$ in S, the syntax $\langle \mathrm{id}; \mathfrak{N}\rangle$ has a cluster-point in S.*

(v) *For every basis $\mathfrak{B}$ of closed sets in S and every filter-base $\mathfrak{N} \subset \mathfrak{B}$, the syntax $\langle \mathrm{id}; \mathfrak{N}\rangle$ has a cluster-point in S.*

(vi) *There exists a basis $\mathfrak{B}$ of closed sets in S such that for every filter-base $\mathfrak{N} \subset \mathfrak{B}$, the syntax $\langle \mathrm{id}; \mathfrak{N}\rangle$ has a cluster-point in S.*

(vii) *Every syntax $\langle f; \mathfrak{N}\rangle$ of points of S has a subsyntax which converges to a point of S.*

(viii) *Every decided syntax of points of S converges to a point of S.*

Proof. (i) $\rightarrow$ (ii). Let $\langle f; \mathfrak{N}\rangle$ be a syntax of points of a compact space S. The set of all images $f(N)$, $N \in \mathfrak{N}$ has the finite intersection property, and therefore so has the set of closures $\overline{f(N)}$, $N \in \mathfrak{N}$. Since S is compact, there exists a point p common to all these closures. For each $N \in \mathfrak{N}$, $p \in \overline{f(N)}$, so for each neighborhood U of p the intersection $U \cap f(N)$ is non-empty, and p is a cluster-point of $\langle f; \mathfrak{N}\rangle$.

(ii) $\rightarrow$ (iii) $\rightarrow$ (iv) $\rightarrow$ (v) $\rightarrow$ (vi). Obvious.

(vi) $\rightarrow$ (i). Let $\mathfrak{A}$ be a family of closed subsets of S having the finite intersection property. Define $\mathfrak{A}'$ to be the family of all sets which are intersections of finitely many members of $\mathfrak{A}$; this too has the finite intersection property. Let $\mathfrak{N}$ consist of all members of the basis $\mathfrak{B}$ which contain a set $A' \in \mathfrak{A}'$; this is a filter-base, and by (vi) the syntax $\langle \mathrm{id}; \mathfrak{N}\rangle$ has a cluster-point p in S. For each N in $\mathfrak{N}$, every neighborhood of p meets id $N = N$, so $p \in \bar{N} = N$. Thus p is in all sets $N \in \mathfrak{N}$. If $A \in \mathfrak{A}$, it is the intersection of all sets of $\mathfrak{B}$ which contain A. But all these sets are in $\mathfrak{N}$, hence contain p, and thus $p \in A$.

From (15), (ii) $\rightarrow$ (viii); from (14), (viii) $\rightarrow$ (vii). From (8) and (9), (vii) $\rightarrow$ (ii). This completes the proof.

Remark 1. While (ii) is a useful consequence of compactness, it is undesirable as a test for compactness, since the family of all syntaxes of points of S is as numerous as the class of all sets. This is the principal reason for including (iii), (iv), (v), and (vi); the cardinal number of syntaxes involved in these criteria can be estimated in terms of the cardinality of S, the last being the least. (Criteria (v) and (vi) were suggested by the referee.)

7. Tychonoff's Theorem. Let A be a non-empty set, and for each $\alpha \in A$ let S_α be a topological space. The cartesian product $\prod_\alpha S_\alpha$ of these S_α is by definition the set of all functions ϕ on A such that for each $\alpha \in A$, $\phi(\alpha) \in S_\alpha$. For each element ϕ of $\prod_\alpha S_\alpha$ and each $\alpha \in A$, the value of ϕ at α is called the *component* of ϕ in S_α and denoted by ϕ_α. The product is topologized by defining neighborhoods to be cartesian products $\prod_\alpha U_\alpha$ in which for finitely many α, U_α is a neighborhood in S_α, and for all other $\alpha \in A$, $U_\alpha = S_\alpha$; the product thus topologized is the "topological product" of the S_α.

If $\langle f; \mathfrak{N} \rangle$ is a syntax of points of $\prod_\alpha S_\alpha$ and for each α in A the component f_α of f is understood to be the function $(f_\alpha(x) \mid x \in D_f)$, then $\langle f_\alpha; \mathfrak{N} \rangle$ is a syntax of points of S_α. If $\langle f; \mathfrak{N} \rangle$ is a decided syntax so is $\langle f_\alpha; \mathfrak{N} \rangle$, as we now show. Let Y_α be any set in the range of f_α, and let Y be the cartesian product $\prod_\beta Z_\beta$, where $Z_\alpha = Y_\alpha$ and $Z_\beta = S_\beta$ for $\beta \in A$, $\beta \neq \alpha$. Then $f(x) \in Y$ if and only if $f_\alpha(x) \in Y_\alpha$. There exists an $N \in \mathfrak{N}$ on which the statement $f(x) \in Y$ is invariably true or invariably false; correspondingly the statement $f_\alpha(x) \in Y_\alpha$ is true for all $x \in X$ or false for all $x \in X$, and so $\langle f_\alpha; \mathfrak{N} \rangle$ is a decided syntax.

From this we obtain a proof of Tychonoff's theorem which is essentially a copy of Cartan's **(5)**.

(17) THEOREM. *If for each $\alpha \in A$ the set S_α is a compact topological space, the topological product of the S_α is also compact.*

Let $\langle f; \mathfrak{N} \rangle$ be a decided syntax of points of the product space. For each $\alpha \in A$, $\langle f_\alpha; \mathfrak{N} \rangle$ is a decided syntax of points of S_α, so by (16) it converges to a point p_α of S_α. Let $p = (p_\alpha \mid \alpha \in A)$; this is a point of $\prod_\alpha S_\alpha$. Now let U be a neighborhood $\prod_\alpha U_\alpha$ of p, where for all α in a finite set $B \subset A$ the set U_α is a neighborhood of p_α in S_α, and for all $\alpha \in A - B$, $U_\alpha = S_\alpha$. For $\alpha \in B$, there exists $N_\alpha \in \mathfrak{N}$ such that $f_\alpha(N_\alpha) \subset U_\alpha$. Since $\mathfrak{N}$ is a filter-base, there exists $N \in \mathfrak{N}$ which is contained in all the N_α, $\alpha \in B$. Then $f_\alpha(N) \subset U_\alpha$, $\alpha \in B$, so $f(N) \subset U$. This proves $\langle f; \mathfrak{N} \rangle$ converges to p, and by (16) the topological product is compact.

REFERENCES

1. A. A. Bennett, *Generalized convergence with binary relations*, Amer. Math. Monthly, *32* (1925), 131–134.
2. G. Birkhoff, *Moore-Smith convergence in general topology*, Ann. Math. (2), *38* (1937), 39–60.
3. ———, *An ergodic theorem for general semi-groups*, Proc. Nat. Acad. Sci., *25* (1939), 625–627.
4. N. Bourbaki, *Éléments de mathématique*, II, Livre III, chap. 1 (Paris, Hermann et Cie., Actualités Scientifiques et Industrielles No. 858, 1940).
5. H. Cartan, *Théorie des filtres*, Comptes Rendus de l'Ac. des Sci. (Paris), *205* (1937), 595–598.
6. ———, *Filtres et ultrafiltres*, Comptes Rendus de l'Ac. des Sci. (Paris), *205* (1937), 777–779.
7. J. L. Kelley, *Convergence in topology*, Duke Math. Jour., *17* (1950), 277–283.
8. E. J. McShane, *Partial orderings and Moore-Smith limits*, Amer. Math. Monthly, *59* (1952), 1–11.
9. ———, *Order-preserving maps and integration* (Princeton Univ. Press, Annals of Math. Studies No. 31, 1953).
10. E. H. Moore, *Definition of limit in general integral analysis*, Proc. Nat. Acad. Sci., *1* (1915), 628–632.
11. ———, *General analysis*, vol. I, pt. II, with the cooperation of R. W. Barnard (American Philosophical Society, 1939).
12. E. H. Moore and H. L. Smith, *A general theory of limits*, Amer. J. Math. *44* (1922), 102–121.
13. H. L. Smith, *A general theory of limits*, National Math. Magazine, *12* (1937–38), 371–379.
14. J. W. Tukey, *Convergence and uniformity in general topology* (Princeton, 1940).

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THE SPECTRAL MATRIX AND GREEN'S FUNCTION FOR SINGULAR SELF-ADJOINT BOUNDARY VALUE PROBLEMS

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1. Introduction. Let L denote the formal ordinary differential operator

$$L = p_0(d/dx)^n + p_1(d/dx)^{n-1} + \ldots + p_n,$$

where we assume the p_k are complex-valued functions with $n-k$ continuous derivatives on an open real interval $a < x < b$ ($a = -\infty$, $b = +\infty$, or both may occur), $p_0(x) \neq 0$ on $a < x < b$, and L coincides with its Lagrange adjoint L^+ given by

$$L^+ = (-1)^n(d/dx)^n(\bar{p}_0 \cdot) + (-1)^{n-1}(d/dx)^{n-1}(\bar{p} \cdot) + \ldots + \bar{p}_n.$$

The purpose of this paper is to prove the existence and uniqueness of Green's function for certain self-adjoint boundary value problems associated with L on (a, b) (Theorems 2 and 3 below), and to show how the existence and uniqueness of the spectral matrix occurring in the Parseval equality may be obtained from Green's function (Theorem 5 below). Further we prove the formula which gives the spectral matrix in terms of Green's function (Theorem 5).

The case $n = 2$, p_k real, was treated initially by Weyl **(11)**. Later Titchmarsh **(9)** gave new proofs of the Weyl results using the residue calculus, and obtained a formula relating Green's function and the spectral matrix. Other proofs of the main results in this case have been given by Stone **(8)**, Kodaira **(2)**, Levinson **(4)**, **(5)**, and Yosida **(12)**. Kodaira **(3)** extended Weyl's results to the case where n is even and the p_k are real, using Hilbert space methods. Levitan **(7)** has considered this case also, and has shown the existence of at least one spectral matrix. It is not difficult to see that his proof carries over to the case of the L considered here. In **(1)** we proved, among other things, the uniqueness of the spectral matrix in two important cases. Another proof along the lines of **(5)** has recently been given by Levinson **(6)**. Here we prove the existence and uniqueness of Green's function and the spectral matrix for these two cases by a simple limiting process on self-adjoint boundary value problems on finite sub-intervals of (a, b). The formula relating Green's function to the spectral matrix reduces to that given by Kodaira **(3)** for the cases he considered.

In §2 we state results we require concerning self-adjoint problems in the non-singular case. Section 3 is first devoted to the proof of the compactness of the set of Green's functions $\{G_\delta\}$ associated with self-adjoint boundary value problems on closed bounded sub-intervals δ of (a, b). The essential idea of the proof (Lemma 3) is an outgrowth of our reading the paper by Titchmarsh **(10)** on

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the Laplace operator in the plane. We then consider two cases: the first where no boundary conditions at a or b are required to obtain a self-adjoint boundary value problem; the second where the end-point a is finite and (a, b) can be replaced by $[a, b)$, and a self-adjoint boundary value problem results by imposing boundary conditions at a alone. For $n = 2$, p_k real, the first case corresponds to the situation where L is of the limit point type at a and b, the second where L is of the limit circle type at a and limit point type at b. For our two cases we prove the existence and uniqueness of Green's function, and in §4 we do the same for the spectral matrix, and give the formula relating the two.

2. The spectral matrix and Green's function in the non-singular case. In this section we collect together several well-known facts concerning self-adjoint boundary value problems on a closed bounded sub-interval $\delta = [\bar{a}, \bar{b}]$ of (a, b) and show how the spectral matrix is related to Green's function in this case.

Let $\mathfrak{D}$ denote the set of all complex-valued functions u of class $\mathfrak{L}^2(a, b)$ which have continuous derivatives up to order $n - 1$ on (a, b), $u^{(n-1)}$ is absolutely continuous on every closed sub-interval of (a, b), and Lu is of class $\mathfrak{L}^2(a, b)$. For functions u, v in $\mathfrak{D}$ we have Green's formula

$$\int_y^x (\bar{v}\, Lu - u\overline{Lv}) = [uv](x) - [uv](y), \tag{2.1}$$

where $a < y < x < b$, and $[uv](x)$ is the form in $(u, u', \ldots, u^{(n-1)})$ and $(v, v', \ldots, v^{(n-1)})$ given by

$$[uv](x) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)}(x)(p_{n-m}\,\bar{v})^{(j)}(x). \tag{2.2}$$

We can write (2.2) as

$$[uv](x) = \sum_{j,k=0}^{n-1} B_{jk}(x)\, u^{(k)}(x)\, \bar{v}^{(j)}(x),$$

and it is readily seen that the matrix $B(x) = (B_{jk}(x))$ is non-singular; in fact its determinant is $(p_0(x))^n$. From Green's formula one finds that

$$[uv](x) = -\overline{[vu](x)}.$$

Suppose

$$U_{\delta j}(u) = \sum_{k=1}^{n} M_{\delta jk}\, u^{(k-1)}(\bar{a}) + N_{\delta jk}\, u^{(k-1)}(\bar{b}) = 0 \quad (j = 1, \ldots, n), \tag{2.3}$$

represent n homogeneous self-adjoint (relative to L) boundary conditions for functions $u \in \mathfrak{D}$ on the interval $\delta = [\bar{a}, \bar{b}]$. Here $M_{\delta jk}$ and $N_{\delta jk}$ are complex constants, and the linearly independent conditions (2.3) are self-adjoint if and only if

$$M_\delta\, B^{-1}(\bar{a})\, M_\delta^* = N_\delta\, B^{-1}(\bar{b})\, N_\delta^*,$$

where M_δ, N_δ are matrices with the elements $M_{\delta jk}$, $N_{\delta jk}$ respectively, B^{-1} is the reciprocal of the matrix B of the form (2.2), and the asterisk indicates the con-

jugate transposed matrix (cf. 1). As is known the self-adjoint boundary value problem

$$Lu = lu, \qquad U_{\delta j}(u) = 0 \qquad (j = 1, \ldots, n), \tag{2.4}$$

(l a complex parameter) possesses a real discrete spectrum. Let $\{\chi_{\delta k}\}$ be a complete orthonormal set of eigenfunctions for this problem, and $\{\lambda_{\delta k}\}$ the corresponding eigenvalues.

The inner product and norm in the space $\mathfrak{L}^2(\delta)$ will be denoted by $(\ ,\ )_\delta$ and $\|\ \|_\delta$ respectively, and thus if $u, v \in \mathfrak{D}$,

$$(u, v)_\delta = \int_\delta u\bar{v}, \quad \|u\|_\delta = \left(\int_\delta |u|^2\right)^{\frac{1}{2}}.$$

In case of the interval (a, b), the inner product and norm will be denoted by $(\ ,\ )$ and $\|\ \|$. For $u, v \in \mathfrak{L}^2(\delta)$ one has the Parseval equality

$$(u, v)_\delta = \sum_{k=1}^{\infty} (u, \chi_{\delta k})_\delta \overline{(v, \chi_{\delta k})_\delta}, \tag{2.5}$$

or in the case $u = v$,

$$\|u\|_\delta^2 = \sum_{k=1}^{\infty} |(u, \chi_{\delta k})_\delta|^2. \tag{2.6}$$

Let $s_j = s_j(x, l)$ $(j = 1, \ldots, n)$ be a set of n linearly independent solutions of the equation $Lu = lu$, where l is a complex number, and to be definite, suppose that for some c, $\bar{a} < c < \bar{b}$, we have

$$s_j^{(k-1)}(c, l) = \delta_{jk} \qquad (j, k = 1, \ldots, n), \tag{2.7}$$

δ_{jk} being the Kronecker delta. Then the s_j are such that the functions $s_j^{(k-1)}$ $(j, k = 1, \ldots, n)$ are entire in l. In terms of such a linearly independent set we have

$$\chi_{\delta k}(x) = \sum_{j=1}^{n} r_{\delta kj} s_j(x, \lambda_{\delta k}), \tag{2.8}$$

where the $r_{\delta kj}$ are complex constants. Placing (2.8) into (2.5) and (2.6) we can rewrite these in the forms

$$(u, v)_\delta = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \bar{\psi}_{\delta j}(\lambda)\ \phi_{\delta k}(\lambda)\ d\rho_{\delta jk}(\lambda), \tag{2.9}$$

$$\|u\|_\delta^2 = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \bar{\phi}_{\delta j}(\lambda)\ \phi_{\delta k}(\lambda)\ d\rho_{\delta jk}(\lambda). \tag{2.10}$$

Here

$$\phi_{\delta k}(\lambda) = (u, s_k(\lambda))_\delta, \quad \psi_{\delta j}(\lambda) = (v, s_j(\lambda))_\delta,$$

where $s_k(\lambda)$ is s_k considered as a function of x for the fixed real $l = \lambda$. The matrix $\rho_\delta = (\rho_{\delta jk})$, called the *spectral matrix* associated with the self-adjoint problem (2.4), consists of step functions with jumps at the eigenvalues $\lambda_{\delta k}$ and possesses the properties:

(i) ρ_δ is hermitian,

(ii) $\rho_\delta(\Delta) = \rho_\delta(\lambda) - \rho_\delta(\mu)$ is positive semi-definite if $\lambda > \mu$ ($\Delta = (\mu, \lambda]$) (we say ρ_δ is non-decreasing),

(iii) the total variation of $\rho_{\delta jk}$ is finite on every finite λ-interval.

In fact we have

$$\rho_{\delta jk}(\lambda_{\delta p} + 0) - \rho_{\delta jk}(\lambda_{\delta p} - 0) = \sum_{\lambda_{\delta m} = \lambda_{\delta p}} r_{\delta mj}\, \bar{r}_{\delta mk}.$$

It is further assumed that $\rho_\delta(0) = 0$ and $\rho_\delta(\lambda + 0) = \rho_\delta(\lambda)$; this justifies the term "the" spectral matrix.

We turn now to the relation between the spectral matrix ρ_δ and Green's function $G_\delta = G_\delta(x, y, l)$ for the problem (2.4). The latter function is a continuous function defined on the square $\bar{a} \leqslant x, y \leqslant \bar{b}$ with the properties:

(i) $\partial^k G_\delta / \partial x^k$ $(k = 0, 1, \ldots, n-2)$ exist, are continuous on the square $\bar{a} \leqslant x, y \leqslant \bar{b}$, and $\partial^{n-1} G_\delta / \partial x^{n-1}$, $\partial^n G_\delta / \partial x^n$ are continuous on each of the triangles $\bar{a} \leqslant x \leqslant y \leqslant \bar{b}$ and $\bar{a} \leqslant y \leqslant x \leqslant \bar{b}$,

(ii) $$\frac{\partial^{n-1} G_\delta}{\partial x^{n-1}}(y + 0, y, l) - \frac{\partial^{n-1} G_\delta}{\partial x^{n-1}}(y - 0, y, l) = \frac{1}{p_0(y)}, \qquad \bar{a} < y < \bar{b},$$

(iii) as a function of x, G_δ satisfies the equation $Lu = lu$, if $x \neq y$,

(iv) as a function of x, G_δ satisfies the boundary conditions

$$U_{\delta j}(u) = 0 \quad (j = 1, \ldots, n) \text{ for } \bar{a} < y < \bar{b}.$$

Further, since the problem (2.4) is self-adjoint, if $\Im l \neq 0$ ($\Im$ = imaginary part of) no function in $\mathfrak{L}^2(\delta)$, except the zero function, satisfies $Lu = lu$ and $U_{\delta j}(u) = 0$ $(j = 1, \ldots, n)$. Hence the solution of the non-homogeneous problem

$$Lu = lu + f, \quad U_{\delta j}(u) = 0 \qquad (j = 1, \ldots, n),$$

where $f \in \mathfrak{L}^2(\delta)$ $(f \neq 0)$, is given by

$$u(x) = \int_\delta G_\delta(x, y, l)\, f(y)\, dy.$$

Let $\mathfrak{D}_\delta$ be the set of all $u \in \mathfrak{L}^2(\delta)$ for which $u^{(n-1)}$ is absolutely continuous on δ, $Lu \in \mathfrak{L}^2(\delta)$, and $U_{\delta j}(u) = 0$, $(j = 1, \ldots, n)$. Then the operator L_δ defined by $L_\delta u = Lu$ for $u \in \mathfrak{D}_\delta$ is a self-adjoint operator in the Hilbert space $\mathfrak{L}^2(\delta)$, and the integral operator $G_\delta(l)$ defined for all $f \in \mathfrak{L}^2(\delta)$ by

$$G_\delta(l)\, f(x) = \int_\delta G_\delta(x, y, l)\, f(y)\, dy$$

is the resolvent of L_δ, that is, $G_\delta(l) = (L_\delta - l)^{-1}$.

In the following it will be convenient to denote G_δ, considered as a function of x (for fixed y and l) as $G_\delta(\ , y, l)$, and similarly when considered as a function of y alone (for fixed x and l) we denote it by $G_\delta(x, \ , l)$. Also we define H_δ by

$$H_\delta(x, y, l) = G_\delta(x, y, l) - G_\delta(x, y, \bar{l}).$$

We remark that the $(n-1)$ *st* derivative of H_δ with respect to x is continuous at $x = y$.

LEMMA 1. *If* $\Im l \neq 0$ *then*

$$2i\Im l \int_\delta \frac{\partial^j G_\delta}{\partial x^j}(x,t,l)\,\frac{\partial^k G_\delta}{\partial y^k}(t,y,\bar{l})\,dt = \frac{\partial^{j+k} H_\delta}{\partial x^j \partial y^k}(x,y,l), \tag{2.11}$$

for $j, k = 0, 1, \ldots, n-1$.

Proof. If $\Im l \neq 0$, $\Im m \neq 0$, then, as is known,

$$(m-l)\int_\delta G_\delta(t,y,m)\,\bar{G}_\delta(t,x,\bar{l})\,dt = G_\delta(x,y,m) - \bar{G}_\delta(y,x,\bar{l}). \tag{2.12}$$

This is the equation satisfied by the kernels of the resolvent operators $G_\delta(l)$, $G_\delta(m)$. A proof results by applying Green's formula (2.1) to the functions $u = G_\delta(\ ,y,m)$, $v = G_\delta(\ ,x,\bar{l})$ on the intervals $[\bar{a}, y-0]$, $[y+0, x-0]$, $[x+0, \bar{b}]$ and adding. Choosing $m = l$ in (2.12) we have

$$\bar{G}_\delta(y,x,\bar{l}) = G_\delta(x,y,l),$$

and as a consequence we have from (2.12) with $m = \bar{l}$,

$$2i\Im l \int_\delta G_\delta(x,t,l)\,G_\delta(t,y,\bar{l})\,dt = G_\delta(x,y,l) - G_\delta(x,y,\bar{l}), \tag{2.13}$$

which is (2.11) for $j = k = 0$. Using the differentiability properties of G_δ it is now obvious that (2.11) follows from (2.13), thus proving the lemma.

LEMMA 2. *The spectral matrix* $\rho_\delta = (\rho_{\delta jk})$ *satisfies the identity*

$$2i\Im l \int_{-\infty}^{\infty} \frac{d\rho_{\delta jk}(\lambda)}{|\lambda - l|^2} = \frac{\partial^{j+k-2} H_\delta}{\partial x^{j-1}\partial y^{k-1}}(c,c,l), \tag{2.14}$$

where $\Im l \neq 0$, *and* $j, k = 1, \ldots, n$.

Proof. Since

$$L\chi_{\delta m} = \lambda_{\delta m}\chi_{\delta m} = l\chi_{\delta m} + (\lambda_{\delta m} - l)\,\chi_{\delta m},$$

we have

$$\chi_{\delta m}(x) = (\lambda_{\delta m} - l)\int_\delta G_\delta(x,t,l)\,\chi_{\delta m}(t)\,dt,$$

or

$$\bar{\chi}_{\delta m}(y) = (\lambda_{\delta m} - \bar{l})\int_\delta G_\delta(t,y,\bar{l})\,\bar{\chi}_{\delta m}(t)\,dt.$$

It follows from this that

$$\bar{\chi}^{(k)}_{\delta m}(y) = (\lambda_{\delta m} - \bar{l})\int_\delta \frac{\partial^k G_\delta}{\partial y^k}(t,y,\bar{l})\,\bar{\chi}_{\delta m}(t)\,dt,$$

for $k = 0, 1, \ldots, n-1$. Therefore the mth Fourier coefficient of $\partial^k G_\delta(\ ,y,\bar{l})/\partial y^k$ with respect to the set $\{\chi_{\delta m}\}$ is

$$\bar{\chi}^{(k)}_{\delta m}(y)/(\lambda_{\delta m} - \bar{l}).$$

From the Parseval relation (2.5) applied to the functions $u = \partial^k G_\delta(\ \ , y, \bar{l})/\partial y^k$ and $v = \partial^j G_\delta(\ \ , x, \bar{l})/\partial y^j$ we obtain

$$\int_\delta \frac{\partial^k G_\delta}{\partial y^k}(t, y, \bar{l})\,\frac{\partial^j \bar{G}_\delta}{\partial y^j}(t, x, \bar{l})\,dt = \int_\delta \frac{\partial^j G_\delta}{\partial x^j}(x, t, l)\,\frac{\partial^k G_\delta}{\partial y^k}(t, y, \bar{l})\,dt = \sum_{m=1}^{m} \frac{\chi_{\delta m}^{(j)}(x)\,\bar{\chi}_{\delta m}^{(k)}(y)}{|\lambda_{\delta m} - l|^2},$$

and in view of Lemma 1 there results

$$2i\Im l \sum_{m=1}^{m} \frac{\chi_{\delta m}^{(j)}(x)\,\bar{\chi}_{\delta m}^{(k)}(y)}{|\lambda_{\delta m} - l|^2} = \frac{\partial^{j+k} H_\delta}{\partial x^j \partial y^k}(x, y, l). \tag{2.15}$$

Using (2.8) we may write, for $\Delta = (\mu, \lambda]$,

$$\sum_{\lambda_{\delta m} \in \Delta} \chi_{\delta m}^{(j)}(x)\,\bar{\chi}_{\delta m}^{(k)}(y) = \int_\Delta \sum_{p,q=1}^{n} s_p^{(j)}(x, \lambda)\,\bar{s}_q^{(k)}(y, \lambda)\,d\rho_{\delta pq}(\lambda),$$

and therefore (2.15) yields

$$2i\Im l \int_{-m}^{m} \sum_{p,q=1}^{n} s_p^{(j)}(x, \lambda)\,\bar{s}_q^{(k)}(y, \lambda)\,|\lambda - l|^{-2}\,d\rho_{\delta pq}(\lambda) = \frac{\partial^{j+k} H_\delta}{\partial x^j\,\partial y^k}(x, y, l).$$

Setting $x = y = c$ in this formula and recalling (2.7) we obtain (2.14), as desired. We are indebted to the referee for pointing out that $H_\delta/2i\Im l$ is the reproducing kernel of the class of all functions satisfying the boundary conditions $U_{\delta j}(u) = 0$ with a norm given by

$$\|u\|^2 = \int_\delta |Lu - lu|^2\,dx.$$

Formula (2.15) follows directly from this. The equation $u = G_\delta(l)(L - l)u$ expresses the reproducing property of $H_\delta/2i\Im l$ in this class.

3. Green's function for singular cases. Here we establish the existence and uniqueness of Green's functions for two self-adjoint boundary value problems on (a, b). Indeed we show that these functions are the limit (uniform in any finite (x, y, l)-region, $\Im l \neq 0$) of Green's functions constructed for self-adjoint problems (2.4) on closed bounded sub-intervals δ of (a, b).

Let $\delta_0 = [a_0, b_0]$ be any closed bounded interval interior to (a, b). Then, as is known, there exists a *fundamental solution* $K_0 = K_0(x, y)$ of the equation $Lu = 0$ on δ_0. This function exists on the square $a_0 \leqslant x, y \leqslant b_0$, and enjoys all the properties of a Green's function for $Lu = 0$ on δ_0 except that it need not satisfy a set of self-adjoint boundary conditions. (Green's function for $Lu = 0$ on δ_0 need not exist, but a fundamental solution always exists.) In order to exhibit the special properties of a fundamental solution we give a construction for one such. Let s_{0j} $(j = 1, \ldots, n)$ denote the set of n linearly independent solutions of $Lu = 0$ given by

$$s_{0j}(x) = s_j(x, 0),$$

satisfying (2.7) with $l = 0$. Using Green's formula one finds that $[s_{0j}\, s_{0k}](x)$ is a constant $[s_{0j}\, s_{0k}]$, independent of x. Let S denote the matrix with element $[s_{0j}\, s_{0k}]$ in the jth row and kth column. From the non-degeneracy of the form $[uv](x)$ and the linear independence of the s_{0j} it follows that the matrix S is non-singular. Moreover we have S is skew-hermitian, since

$$[s_{0j}\, s_{0k}] = -\overline{[s_{0k}\, s_{0j}]}.$$

Let $S^{-1} = (S^{-1}_{jk})$ denote the inverse matrix to S; it is also skew-hermitian. Define K_0 by

$$K_0(x, y) = \bar{K}_0(y, x) = (\tfrac{1}{2}) \sum_{j,k=1}^{n} S^{-1}_{jk}\, s_{0k}(x)\, \bar{s}_{0j}(y) \qquad (x \geqslant y). \tag{3.1}$$

It is an easy task to check that this function is a fundamental solution for $Lu = 0$ on any $\delta_0 \subset (a, b)$, and we shall always mean by K_0 this function which is defined on the whole square $a < x, y < b$.

Suppose $\delta_1 = [a_1, b_1]$ is any other closed bounded sub-interval of (a, b) containing δ_0 properly. Let μ be any real-valued function of class C^∞ on $a < x < b$ such that $\mu(x) = 1$ for $x \in \delta_0$ and $\mu(x) = 0$ when x is outside

$$\tfrac{1}{2}(a_0 + a_1) \leqslant x \leqslant \tfrac{1}{2}(b_0 + b_1).$$

Then define J_1 by

$$J_1(x, y) = \mu(x)\, \mu(y)\, K_0(x, y).$$

Clearly $J_1(x, y) = \bar{J}_1(y, x)$. The basis for our results on Green's function is the following representation of G_δ in terms of J_1.

LEMMA 3. *If $\delta \supset \delta_1 \supset \delta_0$ then for $x \in \delta_0$, $y \in \delta$, and $\Im l \neq 0$,*

$$G_\delta(x, y, l) = J_1(x, y) + \int_{\delta_1} G_\delta(t, y, l)\{\overline{lJ_1}(t, x) - \overline{L_t J_1}(t, x)\}\, dt, \tag{3.2}$$

where L_t denotes L applied to $J_1(\ \ , x)$ for fixed x.

Proof. The function $u = G_\delta(\ \ , y, l) - J_1(\ \ , y)$ for $y \in \delta_0$ is in the set $\mathfrak{D}_\delta$, and hence $u = G_\delta(l)(L_t - l)u$. Using $(L_t - l)\, G_\delta(t, y, l) = 0$ for $t \neq y$, the symmetry $G_\delta(x, y, l) = \bar{G}_\delta(y, x, \bar{l})$, and $J_1(\ \ , y) = 0$ for y outside δ_1 we see (3.2) results.

Since G_δ is unique it may seem a paradox that it can be expressed in terms of the function J_1 which was defined by an almost arbitrary function μ. This "dilemma" disappears when one observes that (3.2) may be written as

$$\begin{aligned} G_\delta(x, y, l) = K_0(x, y) &+ \int_{\delta_0} K_0(x, t)\, l\, G_\delta(t, y, l)\, dt \\ &+ [G_\delta(\ \ , y, l)\, K_0(\ \ , x)](a_0) - [G_\delta(\ \ , y, l)\, K_0(\ \ , x)](b_0), \end{aligned} \tag{3.3}$$

when $x, y \in \delta_0$, and here the values of J_1, for x, y not on the square $a_0 \leqslant x, y \leqslant b_0$ do not enter at all. Now (3.3) results by splitting the integration range in (3.2) into the two ranges δ_0 and $\delta_1 - \delta_0$ and using Green's formula to evaluate the integral over $\delta_1 - \delta_0$. On δ_0, of course $L_t J_1(t, x) = 0$, $x \neq t$.

LEMMA 4. *The set of functions $\{G_\delta\}$ is uniformly bounded and equicontinuous on every compact (x, y, l)-region where $\Im l \neq 0$.*

Proof.[1] From Lemma 3 we have, using the Schwarz inequality, if $\delta \supset \delta_1 \supset \delta_0$ and $x, y \in \delta_0$,

$$|G_\delta(x, y, l)| \leqslant M_0 + \|G_\delta(\ , y, l)\|_{\delta_1} \|\bar{l}\, J_1(\ , x) - L_t J_1(\ , x)\|_{\delta_1}, \tag{3.4}$$

where

$$M_0 = \max |K_0(x, y)| \qquad (x, y \in \delta_0).$$

The uniform boundedness of the G_δ for $\delta \supset \delta_1$ on the square $a_0 \leqslant x, y \leqslant b_0$, and l ranging over some compact set Λ with $\Im l \neq 0$ will follow from (3.4) once we have shown that $\|G_\delta(\ , y, l)\|_{\delta_1}$ is bounded uniformly for $y \in \delta_0$, $l \in \Lambda$. However this follows from the fact that the resolvent $G_\delta(l)$ is a bounded operator with bound not exceeding $|\Im l|^{-1}$, that is, if $u = G_\delta(l)f$, where $f \in \mathfrak{L}^2(\delta)$, then

$$\|u\|_\delta \leqslant |\Im l|^{-1} \|f\|_\delta. \tag{3.5}$$

Applying this to $u = G_\delta(\ , y, l) - J_1(\ , y)$ for $y \in \delta_0$ we see that

$$\|G_\delta(\ , y, l)\|_\delta \leqslant |\Im l|^{-1} \|L_t J_1(\ , y) - l J_1(\ , y)\|_{\delta_1} + \|J_1(\ , y)\|_{\delta_1}, \tag{3.6}$$

and this inequality implies the uniform boundedness of $\|G_\delta(\ , y, l)\|_{\delta_1}$ for $y \in \delta_0$, $l \in \Lambda$.

From (3.2) it follows that[2]

$$\frac{\partial G_\delta}{\partial x}(x, y, l) = \frac{\partial K_0}{\partial x}(x, y) + \int_{\delta_1} G_\delta(t, y, l) \frac{\partial}{\partial x}\{l \bar{J}_1(t, x) - \overline{L_t J_1}(t, x)\}\, dt,$$

and using the same argument as above we see $\partial G_\delta / \partial x$ is uniformly bounded for $a_0 \leqslant x, y \leqslant b_0$, $l \in \Lambda$. The symmetry $G_\delta(x, y, l) = \bar{G}_\delta(y, x, \bar{l})$ implies the same for $\partial G_\delta / \partial y$. From (2.12) we have

$$\frac{\partial G_\delta}{\partial l}(x, y, l) = \int_\delta G_\delta(t, y, l)\, \bar{G}_\delta(t, x, \bar{l})\, dt,$$

and using the Schwarz inequality and (3.6) we find that $\partial G_\delta / \partial l$ is also uniformly bounded for $a_0 \leqslant x, y \leqslant b_0$, $l \in \Lambda$. The uniform boundedness of all first partial derivatives of G_δ implies the equicontinuity of the set $\{G_\delta\}$ thus completing the proof of the lemma.

Certain important conclusions may now be drawn from Lemmas 3 and 4. Firstly, an application of the Ascoli lemma together with Lemma 4 proves that there exists a sequence of intervals $\delta_m \subset (a, b)$ $(m = 1, 2, \ldots)$, $\delta_m \to (a, b)$, such that the corresponding Green's functions $G_m = G_{\delta_m}$ tend uniformly, on any compact subset of $a < x, y < b$, $\Im l > 0$ (or $\Im l < 0$), to a limit function G. This G is defined for $a < x, y < b$, $\Im l \neq 0$, and, being the uniform limit of continuous functions, is continuous. Since the G_m are analytic in l for $\Im l > 0$ (or $\Im l < 0$),

[1] Strictly speaking our proof is given for the cases $n \geqq 2$, but a slight modification shows the validity for $n = 1$; cf. (3.7) and (3.8) below. If $n = 1$, equicontinuity holds for $x \neq y$.

[2] This relation must be modified in case $n = 1$, cf. footnote 1.

the same holds for G. The relation $G_m(x, y, l) = \bar{G}_m(y, x, \bar{l})$ implies that $G(x, y, l) = \bar{G}(y, x, \bar{l})$.

From (3.2) we have, if $a_0 \leqslant x, y \leqslant b_0$, $\Im l \neq 0$, $\delta_0 \subset \delta_1 \subset \delta$,

$$\frac{\partial^j G_\delta}{\partial x^j}(x, y, l) = \frac{\partial^j K_0}{\partial x^j}(x, y) + \int_{\delta_1} G_\delta(t, y, l) \frac{\partial^j}{\partial x^j}\{l\bar{J}_1(t, x) - \overline{L_t J_1}(t, x)\} dt, \tag{3.7}$$

$$(j = 0, 1, \ldots, n-1).$$

Recall that since $\partial^{n-1}G_\delta/\partial x^{n-1}$ and $\partial^{n-1}K_0/\partial x^{n-1}$ have the same discontinuity at $x = y$, their difference is continuous there. Moreover, from (3.7), if $x \neq y$,

$$\frac{\partial^n G_\delta}{\partial x^n}(x, y, l) = \frac{\partial^n K_0}{\partial x^n}(x, y) + \frac{l\, G_\delta(x, y, l)}{p_0(x)} + \int_{\delta_1} G_\delta(t, y, l) \frac{\partial^n}{\partial x^n}\{l\bar{J}_1(t, x) - \overline{L_t J_1}(t, x)\} dt. \tag{3.8}$$

Observing (3.2) with $\delta = \delta_m$, and letting $m \to \infty$, we obtain

$$G(x, y, l) = K_0(x, y) + \int_{\delta_1} G(t, y, l)\{l\bar{J}_1(t, x) - \overline{L_t J_1}(t, x)\} dt,$$

and therefore the partial derivatives $\partial^j G/\partial x^j$ exist and

$$\frac{\partial^j G}{\partial x^j}(x, y, l) = \frac{\partial^j K_0}{\partial x^j}(x, y) + \int_{\delta_1} G(t, y, l) \frac{\partial^j}{\partial x^j}\{l\bar{J}_1(t, x) - \overline{L_t J_1}(t, x)\} dt,$$

$$(j = 0, 1, \ldots, n-1;\ a_0 \leqslant x, y \leqslant b_0;\ \Im l \neq 0),$$

$$\frac{\partial^n G}{\partial x^n}(x, y, l) = \frac{\partial^n K_0}{\partial x^n}(x, y) + \frac{l\, G(x, y, l)}{p_0(x)} + \int_{\delta_1} G(t, y, l) \frac{\partial^n}{\partial x^n}\{l\bar{J}_1(t, x) - \overline{L_t J_1}(t, x)\} dt \quad (x \neq y). \tag{3.9}$$

From (3.9) we infer that as a function of x, G satisfies $Lu = lu$, provided $x \neq y$. (This may perhaps be seen better by observing the relation (3.3) with G_δ replaced by G.) Further, $\partial^{n-1}G/\partial x^{n-1}$ has the same jump at $x = y$ as $\partial^{n-1}K_0/\partial x^{n-1}$, namely,

$$\frac{\partial^{n-1} G}{\partial x^{n-1}}(y + 0, y, l) - \frac{\partial^{n-1} G}{\partial x^{n-1}}(y - 0, y, l) = \frac{1}{p_0(y)}.$$

Since the right sides of (3.7) and (3.8) with $\delta = \delta_m$ tend (as $m \to \infty$) to the right sides of the corresponding formulas in (3.9) we see that

$$\frac{\partial^j G_m}{\partial x^j} \to \frac{\partial^j G}{\partial x^j} \qquad (j = 0, 1, \ldots, n), \tag{3.10}$$

uniformly on any compact (x, y, l) region where $\Im l \neq 0$, and provided $x \neq y$ when $j = n - 1, n$. The relations $G_\delta(x, y, l) = \bar{G}_\delta(y, x, \bar{l})$ and $G(x, y, l) = \bar{G}(y, x, \bar{l})$ imply that

$$\frac{\partial^j G_m}{\partial y^j} \to \frac{\partial^j G}{\partial y^j} \qquad (j = 0, 1, \ldots, n),$$

under the same conditions that (3.10) is valid. Returning to (3.7), (3.8), and (3.9) it is easy to see that the mixed derivatives $\partial^{j+k}G/\partial x^j\,\partial y^k$ $(j, k = 0, 1, \ldots, n-1)$ all exist, and

$$\frac{\partial^{j+k}G_m}{\partial x^j\partial y^k} \to \frac{\partial^{j+k}G}{\partial x^j\partial y^k} \qquad (j, k = 0, 1, \ldots, n-1),$$

uniformly on any compact (x, y, l) region where $\Im l \neq 0$ and $x \neq y$ if j or k is $n-1$.

It follows from (3.6) that there exists a constant c_1 (depending on δ_0 and δ_1 only) such that

$$\|G_\delta(\ \ , y, l)\|_\delta \leqslant c_1|\Im l|^{-1}\,(2|l| + 1) \qquad (y \in \delta_0).$$

But $\|G_\delta(\ \ , y, l)\|_{\bar\delta} \leqslant \|G_\delta(\ \ , y, l)\|_\delta$ for $\bar\delta \subset \delta$, and letting first $\delta \to (a, b)$ through the sequence δ_m and then $\bar\delta \to (a, b)$ we infer, for any fixed (y, l), $\Im l \neq 0$, that $G(\ \ , y, l) \in \mathfrak{L}^2(a, b)$. This implies that for fixed (x, l), $\Im l \neq 0$, $G(x, \ \ , l) \in \mathfrak{L}^2(a, b)$. If $f \in \mathfrak{L}^2(a, b)$ then the integral

$$\int_a^b G(x, t, l)\,f(t)\,dt \qquad (a < x < b,\ \Im l \neq 0),$$

converges absolutely (and uniformly for x in any finite sub-interval of (a, b)), defines a function v, and using the properties developed above for G it is not difficult to see that v has continuous derivatives up to order $n-1$, $v^{(n-1)}$ is absolutely continuous on every closed sub-interval of (a, b), and

$$Lv = lv + f.$$

For example, to prove the existence and continuity of v', one first shows by means of (3.5) applied to $u = (\partial/\partial x)(G_\delta(x, \ \ , l) - J_1(x, \ \ ))$ that $\|\partial G(x, \ \ , l)/\partial x\|$ is bounded for fixed l, $\Im l \neq 0$, uniformly for x on any finite sub-interval of (a, b). Thus the integral

$$\int_a^b \frac{\partial G}{\partial x}\,(x, t, l)\,f(t)\,dt$$

converges uniformly for x on any finite sub-interval or (a, b), and hence represents a continuous function on (a, b) which is easily verified to be v'.

From (3.5) for $\delta = \delta_m$ we obtain, letting $m \to \infty$,

$$\|v\| \leqslant |\Im l|^{-1}\,\|f\|,$$

and since $Lv = lv + f$, we have also

$$\|Lv\| \leqslant (1 + |l|/|\Im l|)\|f\|.$$

In short v belongs to the class of functions $\mathfrak{D}$ defined at the beginning of §2, and $Lv = lv + f$. We summarize our information concerning G in the following statement.

THEOREM 1. *Let G be the limit of any convergent sequence $\{G_m\}$ of the set $\{G_\delta\}$ of Green's functions associated with given self-adjoint boundary value problems*

$$Lu = lu, \quad U_{\delta j}(u) = 0 \qquad (j = 1, \ldots, n),$$

on closed bounded sub-intervals δ *of* (a, b). *Then* G *is continuous for* $a < x, y < b$, *analytic in* l *for* $\Im l > 0$ (*and* $\Im l < 0$), *and possesses the properties*:

(i) $\partial^k G/\partial x^k$ $(k = 0, 1, \ldots, n-2)$ *exist, are continuous on* $a < x, y < b$, *and* $\partial^{n-1}G/\partial x^{n-1}$, $\partial^n G/\partial x^n$ *are continuous on each of the regions* $x < y$ *and* $y < x$,

(ii) $$\frac{\partial^{n-1}G}{\partial x^{n-1}}(y+0, y, l) - \frac{\partial^{n-1}G}{\partial x^{n-1}}(y-0, y, l) = \frac{1}{p_0(y)}, \qquad a < y < b,$$

(iii) *as a function of* x, G *satisfies* $Lu = lu$ *if* $x \neq y$,

(iv) $$\frac{\partial^{j+k}G_\delta}{\partial x^j \partial y^k} \to \frac{\partial^{j+k}G}{\partial x^j \partial y^k} \qquad (j, k = 0, 1, \ldots, n-1),$$

uniformly on any compact (x, y, l) *region where* $\Im l \neq 0$, *and* $x \neq y$ *if* j *or* k *is* $n-1$,

(v) $$G(x, y, l) = \bar{G}(y, x, \bar{l}),$$

(vi) $$G(x, \quad, l) \in \mathfrak{L}^2(a, b) \qquad (a < x < b).$$

If $f \in \mathfrak{L}^2(a, b)$ *the function* v *defined by*

$$v(x) = \int_a^b G(x, t, l)\, f(t)\, dt \qquad (\Im l \neq 0),$$

is an element of $\mathfrak{D}$ *and*

$$Lv = lv + f.$$

We do not call G a Green's function yet for we have not considered the question of boundary conditions on (a, b). Here we shall consider two cases, which reduce to the most important ones when $n = 2$, and show how G may then be considered as a Green's function.

For Case I we assume that there are no functions $u \in \mathfrak{D}$ other than $u = 0$ satisfying $Lu = iu$ or $Lu = -iu$. Then the problem

$$Lu = lu \qquad (u \in \mathfrak{D}), \tag{3.11}$$

is a self-adjoint boundary value problem on (a, b)—with no boundary conditions required other than $u \in \mathfrak{D}$. Another way of putting this is the following: the operator T defined in the Hilbert space $\mathfrak{L}^2(a, b)$ with domain $\mathfrak{D}$ by

$$Tu = Lu \qquad (u \in \mathfrak{D}),$$

is self-adjoint. For the cases treated by Weyl, Case I corresponds to the situation where L is of the limit point type at a and b. It is known that if $\Im l > 0$ $(\Im l < 0)$ the number of solutions $u \in \mathfrak{D}$ of $Lu = lu$ is equal to the number of solutions of $Lu = iu$ $(Lu = -iu)$, and therefore in Case I the equation $Lu = lu$ has no solutions $u \in \mathfrak{D}$ other than the zero function for any l for which $\Im l \neq 0$.

THEOREM 2. *In Case* I, G *is unique, and indeed*

$$G_\delta \to G \qquad (\delta \to (a, b)), \tag{3.12}$$

uniformly on any compact (x, y, l) region where $\Im l \neq 0$, independent of the choice of the self-adjoint boundary conditions on δ. (G is called Green's function for (3.11) *in Case* I.)

Proof. Let G° be any other function having the same properties as the G of Theorem 1. Then from Theorem 1 the function v° given by

$$v^\circ(x) = \int_a^b G^\circ(x, t, l)\, f(t)\, dt \qquad (\Im l \neq 0),$$

exists for any $f \in \mathfrak{L}^2(a, b)$, and

$$Lv^\circ = lv^\circ + f.$$

Thus

$$L(v^\circ - v) = l(v^\circ - v),$$

and this implies, in Case I, that $v^\circ = v$. One readily infers from this, and the symmetry of G and G°, that $G = G^\circ$. This proves the uniqueness of G, and that every convergent sequence of $\{G_k\}$ tends to G, thus giving (3.12).

Before going to Case II we interpret our results in terms of the operator T. Let $G(l)$ denote the operator with domain $\mathfrak{L}^2(a, b)$ defined by

$$G(l)\, f(x) = \int_a^b G(x, t, l)\, f(t)\, dt.$$

It is an integral operator of Carleman type, and

$$G(l) = (T - l)^{-1}. \tag{3.13}$$

Indeed Theorem 1 shows that $G(l)f \in \mathfrak{D}$ for every $f \in \mathfrak{L}^2(a,b)$, and $(L-l)G(l)f=f$. Conversely, let $u \in \mathfrak{D}$, and put $(L - l)u = f$. Then clearly $w = u - G(l)f$ is in $\mathfrak{D}$, and $(L - l)w = (L - l)u - (L - l)G(l)f = f - f = 0$. Thus $w = 0$, or $u = G(l)f$, and thereby $G(l)(L - l)u = u$, proving (3.13).

For *Case* II we consider a half-closed interval $[a, b)$, where a is finite. We assume that the coefficients p_k in L are of class C^{n-k} on $[a, b)$, and $p_0(x) \neq 0$ on $[a, b)$. Let $\mathfrak{D}$ now represent those $u \in \mathfrak{L}^2([a, b))$ with continuous derivatives up to order $n - 1$ on $[a, b)$, $u^{(n-1)}$ absolutely continuous on every closed sub-interval of $[a, b)$ of the form $[a, \tilde{b}]$, $a < \tilde{b} < b$, and $Lu \in \mathfrak{L}^2([a, b))$. Suppose

$$U_j(u) = \sum_{k=1}^{n} M_{jk}\, u^{(k-1)}(a) = 0 \qquad (j = 1, \ldots, \omega), \tag{3.14}$$

is a set of ω linearly independent boundary conditions for functions $u \in \mathfrak{D}$. In Case II we assume that the problem for $u \in \mathfrak{D}$,

$$Lu = lu, \qquad U_j(u) = 0 \qquad (j = 1, \ldots, \omega), \tag{3.15}$$

is a self-adjoint boundary value problem on $[a, b)$. In other words

$$(Lu, v) = (u, Lv)$$

holds for every $u, v \in \mathfrak{D}$ which satisfy the boundary conditions (3.14), and there are no functions $u \in \mathfrak{D}$ (other than $u = 0$) satisfying (3.14) and $Lu = iu$ or

$Lu = -iu$. The latter statement is true if and only if there are no $u \in \mathfrak{D}$ satisfying (3.14) (other than the zero function) and $Lu = lu$ for any l for which $\Im l \neq 0$. In a previous work **(1)** we have shown that Case II can only arise when $n = 2\omega$. In the situation treated by Weyl ($n = 2$) Case II occurs when L is of the limit circle type at a and limit point type at b.

Let $\mathfrak{D}^\circ$ denote the set of all $u \in \mathfrak{D}$ satisfying the boundary conditions (3.14). Then to say that the problem (3.15) is self-adjoint is equivalent to saying that the operator T° defined in $\mathfrak{L}^2([a, b))$ by

$$T^\circ u = Lu \qquad (u \in \mathfrak{D}^\circ),$$

is a self-adjoint operator.

In Case II we consider closed bounded sub-intervals δ of $[a, b)$ of the form $\delta = [a, \bar{b}]$, $a < \bar{b} < b$, and add to the set (3.14) any $n - \omega = \omega$ linearly independent boundary conditions $U_{\delta k}(u) = 0$ such that the combined set

$$U_j(u) = 0, \quad U_{\delta k}(u) = 0 \qquad (j, k = 1, \ldots, \omega), \tag{3.16}$$

is a self-adjoint set on δ. Let G_δ represent Green's function for $\Im l \neq 0$ of the self-adjoint problem

$$Lu = lu, \quad U_j(u) = 0, \quad U_{\delta k}(u) = 0 \qquad (j, k = 1, \ldots, \omega).$$

Then it is clear that Lemma 4 and Theorem 1 remain valid for the set $\{G_\delta\}$. If G is the limit of any convergent sequence $\{G_m\}$ of $\{G_\delta\}$, then since every G_δ (considered as a function of x) satisfies the conditions $U_j(u) = 0$ ($j = 1, \ldots, \omega$) for fixed y, $a < y < \bar{b}$, we have that G, as a function of x, satisfies $U_j(u) = 0$ ($j = 1, \ldots, \omega$) each fixed y, $a < y < b$. We infer from this that if $f \in \mathfrak{L}^2([a, b))$ the function

$$v(x) = \int_a^b G(x, t, l)\, f(t)\, dt \qquad (\Im l \neq 0),$$

satisfies the boundary conditions $U_j(u) = 0$ ($j = 1, \ldots, \omega$). From these considerations it is easy to show, just as in Theorem 2, that every convergent sequence of $\{G_\delta\}$ must tend to the same limit, and hence $G_\delta \to G$ as $\delta = [a, \bar{b}] \to [a, b)$, i.e., as $\bar{b} \to b$. Moreover G is unique; it is the only G with the properties listed in Theorem 1 which satisfies the conditions $U_j(u) = 0$ ($j = 1, \ldots, \omega$).

THEOREM 3. *In Case* II *G is unique, and $G_\delta \to G$, as $\bar{b} \to b$, uniformly on any compact (x, y, l) region where $\Im l \neq 0$, independent of the choice of the boundary conditions $U_{\delta k}(u) = 0$ ($k = 1, \ldots, \omega$) on δ. (In Case* II *G is called Green's function for the problem* (3.15)).

In terms of the operator T° defined above, we have shown that the operator $G(l)$ defined by

$$G(l)\, f(x) = \int_a^b G(x, t, l)\, f(t)\, dt \qquad (f \in \mathfrak{L}^2([a, b))),$$

is an integral operator of Carleman type, and

$$G(l) = (T^\circ - l)^{-1}.$$

4. The spectral matrix for singular cases. We are now in a position to exploit the relation (2.14) in Lemma 2, which connects the spectral matrix ρ_δ with Green's function G_δ. For brevity we put

$$P_{\delta jk}(l) = \frac{\partial^{j+k-2} H_\delta}{\partial x^{j-1} \partial y^{k-1}}(c, c, l) \qquad (j, k = 1, \ldots, n),$$

where we recall

$$H_\delta(x, y, l) = G_\delta(x, y, l) - G_\delta(x, y, \bar{l}).$$

Also we write

$$P_{jk}(l) = \frac{\partial^{j+k-2} H}{\partial x^{j-1} \partial y^{k-1}}(c, c, l) \qquad (j, k = 1, \ldots, n),$$

where

$$H(x, y, l) = G(x, y, l) - G(x, y, \bar{l}),$$

and G is the limit of any convergent sequence $\{G_m\}$ of the set $\{G_\delta\}$.

THEOREM 4. *Let $\{G_m\}$ be any convergent sequence of the set $\{G_\delta\}$, and let the corresponding spectral matrices be $\rho_m = (\rho_{mjk})$. Then there exists an hermitian, non-decreasing matrix $\rho = (\rho_{jk})$ whose elements are of bounded variation on every finite λ-interval, such that if $\Delta = (\mu, \lambda]$ is a finite interval whose end-points are continuity points for ρ_{jk}, then*

$$\rho_{mjk}(\Delta) \to \rho_{jk}(\Delta) \qquad (m \to \infty).$$

Further, if $G_m \to G$, then in terms of P_{jk} constructed above,

$$\rho_{jk}(\Delta) = \frac{1}{2\pi i} \lim_{\epsilon \to +0} \int_\mu^\lambda P_{jk}(\sigma + i\epsilon)\, d\sigma. \tag{4.1}$$

Proof. Using the results developed in §3 the proof reduces to the argument given by Levinson **(4)** in the case $n = 2$, p_k real. From (2.14) we have

$$2i\Im l \int_{-\infty}^{\infty} \frac{d\rho_{mjk}(\lambda)}{|\lambda - l|^2} = P_{mjk}(l) \qquad (\Im l \neq 0), \tag{4.2}$$

and for $l = i$,

$$\int_{-\mu}^{\mu} \frac{d\rho_{mjj}(\lambda)}{1 + \lambda^2} \leqslant \int_{-\infty}^{\infty} \frac{d\rho_{mjj}(\lambda)}{1 + \lambda^2} = \frac{P_{mjj}(i)}{2i}, \tag{4.3}$$

since ρ_{mjj} is non-decreasing. The right side of (4.3) is bounded uniformly in m since by Theorem 1 (iv) $P_{mjj} \to P_{jj}$ uniformly on any compact subset Λ of $\Im l > 0$ (or $\Im l < 0$). Therefore, from (4.3), there exists a constant C not depending on m such that

$$\int_{-\mu}^{\mu} d\rho_{mjj}(\lambda) < C(1 + \mu^2) \qquad |\rho_{mjj}(\lambda)| < C(1 + \lambda^2).$$

But $|\rho_{mjk}(\Delta)|^2 \leqslant \rho_{mjj}(\Delta)\, \rho_{mkk}(\Delta)$, and hence the total variation of the ρ_{mjk} on any finite λ-interval is bounded independent of m. By the Helly selection theorem there exists a subsequence of the matrices $\rho_m = (\rho_{mjk})$ tending to a limit matrix

ρ, which is hermitian, non-decreasing, and of bounded variation on every finite λ-interval.

From (4.2) we readily infer that

$$\int_{-\infty}^{\infty} \frac{d\rho_{jk}(\lambda)}{|\lambda - l|^2}$$

converges absolutely. If $\Im l \neq 0$, $\Im l^\circ \neq 0$, then the integral

$$\int_{-\infty}^{\infty} \left(\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l^\circ|^2} \right) d\rho_{mjk}(\lambda) \tag{4.4}$$

tends, as $m \to \infty$ through the subsequence, to

$$\int_{-\infty}^{\infty} \left(\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l^\circ|^2} \right) d\rho_{jk}(\lambda).$$

But (4.4) is just

$$\frac{P_{mjk}(l)}{2i\Im l} - \frac{P_{mjk}(l^\circ)}{2i\Im l^\circ},$$

which, by Theorem 1 (iv), tends to

$$\frac{P_{jk}(l)}{2i\Im l} - \frac{P_{jk}(l^\circ)}{2i\Im l^\circ}.$$

Therefore we see that

$$\frac{P_{jk}(l)}{2i\Im l} - \int_{-\infty}^{\infty} \frac{d\rho_{jk}(\lambda)}{|\lambda - l|^2}$$

is a constant independent of l for $\Im l \neq 0$. The relation (4.1) now follows readily from this. Indeed

$$\begin{aligned}
\lim_{\epsilon \to +0} \int_{\mu}^{\lambda} P_{jk}(\sigma + i\epsilon)\, d\sigma &= 2i \lim_{\epsilon \to +0} \int_{\mu}^{\lambda} \left(\int_{-\infty}^{\infty} \frac{\epsilon\, d\rho_{jk}(\nu)}{(\nu - \sigma)^2 + \epsilon^2} \right) d\sigma \\
&= 2i \int_{-\infty}^{\infty} \lim_{\epsilon \to +0} \left[\tan^{-1}\left(\frac{\lambda - \nu}{\epsilon} \right) - \tan^{-1}\left(\frac{\mu - \nu}{\epsilon} \right) \right] d\rho_{jk}(\nu) \\
&= 2\pi i\, (\rho_{jk}(\lambda) - \rho_{jk}(\mu)),
\end{aligned}$$

provided λ, μ are points of continuity for ρ_{jk}. This proves (4.1).

It is now clear from the relation (4.1) that every convergent subsequence of $\{\rho_{mjk}(\Delta)\}$ must tend to the same limit, and therefore $\rho_{mjk}(\Delta) \to \rho_{jk}(\Delta)$ if the end-points of Δ are continuity points of ρ_{jk}. This completes the proof of Theorem 4.

For Cases I and II we showed that $G_\delta \to G$ as $\delta \to (a, b)$ ($\delta = [a, \bar{b}] \to [a, b)$ in Case II), and from Theorem 4 it is obvious that the corresponding spectral matrices $\rho_\delta = (\rho_{\delta jk})$ satisfy $\rho_\delta(\Delta) \to \rho(\Delta)$, if the end-points of Δ are continuity points of ρ. Further relation (4.1) holds.

THEOREM 5. *In Case* I *or* II *there exists an hermitian, non-decreasing matrix* $\rho = (\rho_{jk})$ *whose elements are of bounded variation on every finite* λ*-interval, and*

which is essentially unique, in the sense that if $\Delta = (\mu, \lambda]$ *is a finite interval whose end-points are continuity points for* ρ_{jk} *then*

$$\rho_{\delta jk}(\Delta) \to \rho_{jk}(\Delta) \qquad \begin{matrix}(\delta \to (a, b) \text{ in Case I,} \\ \delta \to [a, b) \text{ in Case II}).\end{matrix}$$

Further

$$\rho_{jk}(\Delta) = \frac{1}{2\pi i} \lim_{\epsilon \to +0} \int_{\mu}^{\lambda} P_{jk}(\sigma + i\epsilon)\, d\sigma,$$

where

$$P_{jk}(l) = \frac{\partial^{j+k-2} H}{\partial x^{j-1} \partial y^{k-1}}(c, c, l), \quad H(x, y, l) = G(x, y, l) - G(x, y, \bar{l}),$$

and G *is Green's function for Case* I *or Case* II.

In Case I or II we call ρ the *spectral matrix*. If ρ is any limit matrix given by Theorem 4 there always exists a Parseval equality and expansion theorem valid for functions $u \in \mathfrak{L}^2(a, b)$. The proof given by Levinson **(4)** can be carried over to this case; cf. also Levitan **(7)**. Let $\phi = (\phi_1, \ldots, \phi_n)$, $\psi = (\psi_1, \ldots, \psi_n)$ be vector functions of λ, and introduce the inner product

$$(\phi, \psi) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \bar{\psi}_j(\lambda)\, \phi_k(\lambda)\, d\rho_{jk}(\lambda).$$

Since ρ is non-decreasing, $(\phi, \phi) \geqq 0$ and we can define a norm $\| \ \|^*$ by $\|\phi\|^* = (\phi, \phi)^{\frac{1}{2}}$. Let $\mathfrak{H}^*$ be the Hilbert space of all ϕ such that $\|\phi\|^* < \infty$.

THEOREM 6. *Let* ρ *be any limit matrix given by Theorem* 4. *If* $u \in \mathfrak{L}^2(a, b)$ *the vector* $\phi = (\phi_j)$ *where*

$$\phi_j(\lambda) = \int_a^b \bar{s}_j(x, \lambda)\, u(x)\, dx,$$

converges in norm in $\mathfrak{H}^*$, *and*

$$\|\phi\|^* = \|u\| \qquad \text{(Parseval equality)}.$$

In terms of this ϕ,

$$u(x) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} s_j(x, \lambda)\, \phi_k(\lambda)\, d\rho_{jk}(\lambda) \quad \text{(Expansion theorem)}$$

where the integral converges in norm in $\mathfrak{L}^2(a, b)$.

The convergence of ϕ in norm in $\mathfrak{H}^*$ is meant in the sense that the vector $\phi_\delta = (\phi_{\delta j})$, where

$$\phi_{\delta j}(\lambda) = \int_\delta \bar{s}_j(x, \lambda)\, u(x)\, dx,$$

converges strongly in $\mathfrak{H}^*$ to ϕ when $\delta \to (a, b)$. Similarly the expansion theorem is meant in the sense that

$$\int_\Delta \sum_{j,k=1}^{n} s_j(x, \lambda)\, \phi_k(\lambda)\, d\rho_{jk}(\lambda)$$

converges strongly in $\mathfrak{L}^2(a, b)$ to u as $\Delta \to (-\infty, \infty)$.

References

1. Coddington, E. A., *On the spectral representation of ordinary self-adjoint differential operators*, Proc. Nat. Acad. Sci., *38* (1952), 732–737.
2. Kodaira, K., *Eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices*, Amer. J. Math., *71* (1949), 921–945.
3. ———, *On ordinary differential equations of any even order and the corresponding eigenfunction expansions*, Amer. J. Math., *72* (1950), 502–544.
4. Levinson, N., *A simplified proof of the expansion theorem for singular second order linear differential equations*, Duke Math. J., *18* (1951), 57–71.
5. ———, *Addendum to above*, Duke Math. J., *18* (1951), 719–722.
6. ———, *The expansion theorem for singular self-adjoint linear differential operators*, to appear in Annals of Math.
7. Levitan, B. M., *Proof of a theorem on eigenfunction expansions for self-adjoint differential equations*, Doklady Akad. Nauk. USSR, *73* (1950), 651–654.
8. Stone, M. H., *Linear transformations in Hilbert space*, Amer. Math. Soc. Coll. Publication (1932).
9. Titchmarsh, E. C., *Eigenfunction expansions associated with second order differential equations* (Oxford, 1946).
10. ———, *Eigenfunction expansions associated with partial differential equations*, Proc. London Math. Soc., *1* (1951), 1–27.
11. Weyl, H., *Ueber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Annalen, *68* (1910), 220–269.
12. Yosida, K., *On Titchmarsh-Kodaira's formula concerning Weyl-Stone's eigenfunction expansion*, Nagoya Math. J., *1* (1950) 49–58.

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CERTAIN FOURIER TRANSFORMS OF DISTRIBUTIONS: II

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1. Introduction. In an earlier paper **(1)**, published in this journal, a necessary condition was given which the reciprocal of a polynomial without multiple roots must satisfy in order to be a characteristic function. This condition is, however, valid for a wider class of functions since it can be shown **(2**, theorem 2 and corollary to theorem 3**)** that it holds for all analytic characteristic functions. The proof given in **(1)** is elementary and has some methodological interest since it avoids the use of theorems on singularities of Laplace transforms. Moreover the method used in **(1)** yields some additional necessary conditions which were not given in **(1)** and which do not seem to follow easily from the properties of analytic characteristic functions. The purpose of this note is to supplement **(1)** by establishing these conditions and by deriving the results for rational functions which need not be the reciprocals of polynomials without multiple zeros. It should be remarked that Takano **(3)** extended the conditions of **(1)** to the reciprocals of polynomials having multiple roots.

2. Necessary conditions for rational characteristic functions. In this note we derive the following

THEOREM. *If the rational function $\phi(t)$ is a characteristic function then the following three conditions are satisfied:*

(a) *$\phi(t)$ has no poles on the real axis. All the poles and zeros of $\phi(t)$ are either all located on the imaginary axis or they occur in pairs $\pm b + ia$, symmetric with regard to this axis. If $\phi(t)$ is written as the quotient of two polynomials then the degree of its numerator can not exceed the degree of its denominator.*

(b) *If $b + ia$ $(a, b$ real, $a \neq 0$, $b \neq 0)$ is a pole of $\phi(t)$ then $\phi(t)$ has at least one pole $i\alpha$ (α real) such that sgn α = sgn a and $|\alpha| \leqslant |a|$.*

(c) *Let a be the imaginary part of the poles of $\phi(t)$ which are closest to the real axis in the upper (lower) half plane. If $\phi(t)$ has not only the pole ia but also poles $\pm b + ia$ $(a, b$ real $a \neq 0$, $b \neq 0)$ then*

(i) *no pole of the form $b + ia$ can have a higher multiplicity than the pole ia*

(ii) *if the only poles of $\phi(t)$ with imaginary part a are ia and $\pm b + ia$ and if these poles have equal multiplicity s we have*

$$A_{1,s}\, a^s - 2|C_{1,s}|\ (a^2 + b^2)^{\frac{1}{2}s} > 0.$$

Here $A_{1,s}$ is the coefficient of $(1 - it/a)^{-s}$ and $C_{1,s}$ is the coefficient of $(1 - it/(a + ib))^{-s}$, in the expansion of $\phi(t)$ into partial fractions.

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Conditions (a) and (b) correspond to conditions (a) and (b) of **(1)**; the proof is practically identical with the proof given in the earlier paper so that we have now to establish only (c). We shall therefore assume from now on that $\phi(t)$ satisfies (a) and (b). We can then write

$$\phi(t) = \frac{P(it)}{Q(it)} \tag{1}$$

where t is a real variable while $P(z)$ and $Q(z)$ are polynomials in z with real coefficients.

3. Proof of condition (c). We divide the zeros of $Q(it)$ into four groups:

(i) zeros $i\beta_h$ $(h = 1, \ldots, \mu)$ on the positive imaginary axis;

(ii) zeros $-i\alpha_j$ $(j = 1, \ldots, \nu)$ on the negative imaginary axis;

(iii) p pairs of complex zeros in the upper half plane iw_k and $i\bar{w}_k$ where $w_k = c_k + id_k$ $(k = 1, \ldots, p)$;

(iv) n pairs of complex roots in the lower half plane $-iv_m$ and $-i\bar{v}_m$ where $v_m = a_m + ib_m$ $(m = 1, \ldots, n)$.

The quantities $a_m, b_m, c_k, d_k, \alpha_j, \beta_h$ are positive numbers. We denote by q_m, ρ_j, r_k, σ_h the multiplicities of the zeros $v_m, \alpha_j, w_k, \beta_h$ respectively. It is sufficient to prove (c) only for one of the half planes. (See the argument used in **(1)**.) Therefore it is no restriction to suppose that $Q(it)$ has zeros in the lower half plane. The assumptions of condition (c) imply then that $\nu > 0$ and $n > 0$. According to condition (a) of our theorem, the degree of $P(z)$ cannot exceed the degree of $Q(z)$; we obtain therefore from (1) the decomposition of $\phi(t)$ into partial fractions

$$\begin{aligned}\phi(t) = &\sum_{j=1}^{\nu}\sum_{\lambda=1}^{\rho_j}\frac{A_{j,\lambda}}{(1-it/\alpha_j)^\lambda} + \sum_{m=1}^{n}\sum_{\lambda=1}^{q_m}\left[\frac{C_{m,\lambda}}{(1-it/v_m)^\lambda} + \frac{\bar{C}_{m,\lambda}}{(1-it/\bar{v}_m)^\lambda}\right] \\ &+ \sum_{h=1}^{\mu}\sum_{\lambda=1}^{\sigma_h}\frac{B_{h,\lambda}}{(1+it/\beta_h)^\lambda} + \sum_{k=1}^{p}\sum_{\lambda=1}^{r_k}\left[\frac{D_{k,\lambda}}{(1+it/w_k)^\lambda} + \frac{\bar{D}_{k,\lambda}}{(1+it/\bar{w}_k)^\lambda}\right].\end{aligned} \tag{2}$$

If either μ or p should be equal to zero, then the corresponding sum is omitted. From formulae (3.11), (3.12), (3.21), (3.22) of our earlier paper we can compute the frequency function

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\,\phi(t)\,dt.$$

We see that, for $x > 0$,

$$\begin{aligned}f(x) = &\sum_{j=1}^{\nu}\sum_{\lambda=1}^{\rho_j}\frac{A_{j,\lambda}\,\alpha_j^\lambda}{\Gamma(\lambda)}\,x^{\lambda-1}\,e^{-\alpha_j x} \\ &+ \sum_{m=1}^{n} e^{-xa_m}\sum_{\lambda=1}^{q_m}\frac{x^{\lambda-1}}{\Gamma(\lambda)}\,[C_{m,\lambda}\,v_m^\lambda\,e^{-ixb_m} + \bar{C}_{m,\lambda}\,\bar{v}_m^\lambda\,e^{ixb_m}].\end{aligned} \tag{3}$$

A similar expression can be found for $f(x)$ if $x < 0$ and $\mu \neq 0$ or $p \neq 0$ but is not needed in the sequel. If $\mu = p = 0$ then $f(x) = 0$ for $x < 0$.

We choose our notation so that

$$a_1 \leqq a_2 \leqq \ldots \leqq a_n, \tag{4.1}$$
$$\alpha_1 < \alpha_2 < \ldots < \alpha_r.$$

We denote by N the greatest integer such that $a_j = a_1$ for $j = 1, 2, \ldots, N$. Clearly $1 \leqq N \leqq n$ and we have $a_N < a_{N+1}$ in case $N < n$. It is convenient to put

$$t = \max [q_1, q_2, \ldots q_N]. \tag{4.2}$$

We can then write (3) for large positive values of x as

$$f(x) = e^{-xa_1} x^{\rho_1 - 1} \left[\frac{A_{1,\rho_1} \alpha_1^{\rho_1}}{\Gamma(\rho_1)} + o(1) \right] + e^{-xa_1} x^{t-1} [h(x) + o(1)] \tag{5}$$

where

$$h(x) = \sum_{m=1}^{N} \frac{1}{\Gamma(t)} [C_{m,t} v_m^t e^{-ixb_m} + \bar{C}_{m,t} \bar{v}_m^t e^{ixb_m}]. \tag{5.1}$$

The summation is here to be extended over all integers m $(1 \leqq m \leqq N)$ for which $q_m = t$.

Since we assume that conditions (a) and (b) are satisfied we have $\alpha_1 \leqq a_1$. We write

$$s = \begin{cases} \rho_1 & \text{if } \alpha_1 < a_1, \\ \max [\rho_1, t] & \text{if } \alpha_1 = a_1. \end{cases} \tag{6.1}$$

We see then easily from (5) and (5.1) that for large positive values of x,

$$f(x) = e^{-xa_1} x^{s-1} [g(x) + o(1)], \tag{7}$$

where

$$g(x) = \begin{cases} \dfrac{A_{1,\rho_1} \alpha_1^{\rho_1}}{\Gamma(\rho_1)} & \text{if } \alpha_1 < a_1 \text{ or if } \alpha_1 = a_1 \text{ and } t < \rho_1 = s, \\ h(x) & \text{if } \alpha_1 = a_1 \text{ and } \rho_1 < t = s, \\ \dfrac{A_{1,s} \alpha_1^{s}}{\Gamma(s)} + h(x) & \text{if } \alpha_1 = a_1 \text{ and } \rho_1 = t = s. \end{cases} \tag{7.1}$$

We are now ready to establish condition (c). Let us first assume that $a_1 = \alpha_1$ and $\rho_1 < t$ then

$$f(x) = e^{-xa_1} x^{s-1} [h(x) + o(1)].$$

It is seen from (5.1) that $h(x)$ is an almost periodic function without constant term; according to the lemma given in **(1)**, $h(x)$ assumes also negative values. From the almost periodicity of $h(x)$ it follows easily that $f(x)$ must assume negative values for some sufficiently large values of x. This proves part (i) of condition (c).

We next consider the case $a_1 = \alpha_1$ and $\rho_1 = t = s$. Since the function

$$\frac{A_{1,s} \alpha_1^{s}}{\Gamma(s)} + h(x)$$

is almost periodic we conclude from (5.1), (7) and (7.1) that $f(x)$ is nonnegative for positive x if and only if

$$A_{1,s}\alpha_1^s + \sum_{m=1}^{N} [C_{m,s} v_m^s e^{-ixb_m} + \bar{C}_{m,s} \bar{v}_m^s e^{ixb_m}] \geqslant 0 \tag{8.1}$$

for all $x > 0$ (summation extended over integers m for which $q_m = s$).

If we write $C_{m,s} v_m{}^s = R_{m,s} [\cos \theta_{m,s} + i \sin \theta_{m,s}]$ we obtain easily from (8.1)

$$A_{1,s}\alpha_1^s + \sum_{m=1}^{N} 2 R_{m,s} \cos (\theta_{m,s} - b_m x) \geqslant 0 \qquad \text{for } x > 0. \tag{8.2}$$

If in particular $N = 1$ this reduces to

$$A_{1,s}\alpha_1^s + 2 R_{1,s} \cos (\theta_{1,s} - b_1 x) \geqslant 0 \qquad \text{for } x > 0,$$

and therefore

$$A_{1,s}\alpha_1^s - 2 R_{1,s} \geqslant 0. \tag{8.3}$$

If we write here for $R_{1,s} = |C_{1,s} v_1{}^s|$ and for $\alpha_1 = a$ and for $v_1 = ia + b$ we obtain the statement (ii) of condition (c).

It is worthwhile to remark that the above reasoning would still hold for certain irrational meromorphic functions.

References

1. Eugene Lukacs and Otto Szász, *Certain Fourier transforms of distributions*, Can. J. Math., **3** (1951), 140–144.
2. Eugene Lukacs and Otto Szász, *On analytic characteristic functions*, Pacific J. Math., **2** (1952), 615–625.
3. Kinsaku Takano, *Certain Fourier transforms of distributions*, Tohoku Math. J. (2), **3** (1951), 306–315.

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AN INVERSION AND REPRESENTATION THEORY FOR THE LAPLACE INTEGRAL OF ABSTRACTLY-VALUED FUNCTIONS

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1. Introduction. The theory of the Laplace integral of abstractly-valued functions of a real or complex variable has been developed, in the last few years, to an extent that it is almost approaching the degree of completeness enjoyed by the classical theory of the Laplace integral of numerically-valued functions. In certain respects, however, there are still large gaps. One of the gaps occurs in representation theory.

In particular, there are no theorems giving conditions that an abstractedly-valued function be represented as the Laplace integral **(4)** of a function in $B_p([0, \infty); \mathfrak{X})$, the Banach space equivalent of $L_p(0, \infty)$. It is our purpose here to fill this gap.

Since, as in the numerically-valued case, this representation theory is developed in terms of a particular inversion operator for the Laplace integral, the opportunity was offered further to study some inversion operator for the transform. We have grasped this opportunity, and have developed the theory in terms of a certain "real" inversion operator given by the formula

$$\text{I} \qquad L_{\kappa,\tau}[f(\lambda)] = (\kappa e^{2\kappa}/\pi\tau) \int_0^\infty \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}}) f(\kappa(\eta + 1)/\tau)\, d\eta$$

or by the alternative formula, which we shall use occasionally,

$$\text{I}_a \qquad L_{\kappa,\tau}[f(\lambda)] = (2\kappa e^{2\kappa}/\pi\tau) \int_0^\infty \cos(2\kappa\eta) f(\kappa(\eta^2 + 1)/\tau)\, d\eta,$$

where the integrals are Bochner integrals.

The theory of this operator for numerically-valued functions has already appeared in print; see Rooney **(7)**.

The fact that the representation theory is stated in terms of a particular operator is no restriction, since the method is quite general, and will work quite well with any operator for which similar theorems hold in the numerically-valued case.

In §2 of this paper we define a slight generalization of the one-dimensional Bochner integral and prove one or two theorems concerning this generalization. In §3 we prove a lemma concerning the Lebesgue sets of abstractly-valued functions, while §4 is given over to a theorem enabling us to evaluate a singular integral that appears repeatedly in the theory.

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In §5 we prove an inversion theorem for the operator I.

§6 is devoted to certain lemmas preliminary to the representation theory, while in §7 we prove the "Fundamental Theorem" which gives sufficient conditions that the Laplace integral of $L_{\kappa,\tau}[f(\lambda)]$ should tend to $f(\lambda)$ as κ tends to infinity.

In §8, we give necessary and sufficient conditions that a function $f(\lambda)$ be the Laplace integral of a function in $B_p([0, \infty); \mathfrak{X})$, $1 < p < \infty$, where $\mathfrak{X}$ is a reflexive Banach space. In §9 we prove a similar theorem for $B_\infty([0, \infty); \mathfrak{X})$, $\mathfrak{X}$ now uniformly convex.

Lastly, in §10, we give sufficient conditions that a function in $H_1(\alpha; \mathfrak{X})$ be a Laplace integral, and we also have a theorem concerning $H_p(\alpha; \mathfrak{X})$, $p \neq 1$.

Throughout this paper we use the notation of Hille **(4)**. Also, whenever we speak of limits, we mean limits in the strong topology, unless otherwise specified.

2. Improper Bochner integrals. For the inversion theory of the operator $L_{\kappa,\tau}[f(\lambda)]$ we shall use a slightly generalized form of a one dimensional Bochner integral which we shall call the improper Bochner integral.

DEFINITION 2.1. Let $x(\alpha)$ be in $B([\lambda, \omega]; \mathfrak{X})$ for a fixed λ and all $\omega > \lambda$. If

$$\int_\lambda^\omega x(\alpha)\, d\alpha$$

converges to a limit y as $\omega \to \infty$, then we say that y is the *improper Bochner integral* of $x(\alpha)$ over $[\lambda, \infty)$, and we write

$$y = \int_\lambda^{+\infty} x(\alpha)\, d\alpha.$$

We shall prove two theorems; the first giving sufficient conditions for the interchange of the order of integrations when one of the integrals involved is an improper Bochner integral and the other is a Bochner integral, and the second giving sufficient conditions for Bochner and improper Bochner integrals to be equivalent.

Let $E_{\omega,\zeta} = \{(\alpha, \beta) | \, 0 \leqslant \alpha \leqslant \omega; 0 \leqslant \beta \leqslant \zeta\}$.

THEOREM 2.1. *If*

(1) $x(\alpha, \beta)$ *is in* $B(E_{\omega,\zeta}; \mathfrak{X})$ *for a fixed* ζ *and all* $\omega > 0$,

(2) $\int_0^{+\infty} x(\alpha, \beta)\, d\alpha$ *converges uniformly with respect to* β, $0 \leqslant \beta \leqslant \zeta$, *then*

(i) $\int_0^{+\infty} \int_0^\zeta x(\alpha, \beta)\, d\beta\, d\alpha$ *converges,*

(ii) $\int_0^{+\infty} x(\alpha, \beta)\, d\alpha$ *is in* $B([0, \zeta]; \mathfrak{X})$,

(iii) $\int_0^{+\infty} \int_0^\zeta x(\alpha, \beta)\, d\beta\, d\alpha = \int_0^\zeta \int_0^{+\infty} x(\alpha, \beta)\, d\alpha\, d\beta.$

Proof.

(i) It is sufficient to show that for every $\epsilon > 0$, $\omega(\epsilon)$ exists such that

$$\left\| \int_{\omega_1}^{\omega_2} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\| < \epsilon, \qquad \omega_2 > \omega_1 > \omega(\epsilon).$$

By (2), for every $\epsilon > 0$, $\omega_3(\epsilon)$ exists such that

$$\left\| \int_{\omega_1}^{\omega_2} x(\alpha, \beta)\, d\alpha \right\| < \epsilon, \qquad \omega_2 > \omega_1 > \omega_3(\epsilon).$$

Let $\omega(\epsilon) = \omega_3(\epsilon/\zeta)$. Then if $\omega_2 > \omega_1 > \omega(\epsilon)$,

$$\begin{aligned} \left\| \int_{\omega_1}^{\omega_2} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\| &= \left\| \int_0^{\zeta} \int_{\omega_1}^{\omega_2} x(\alpha, \beta)\, d\alpha\, d\beta \right\| \\ &\leqslant \int_0^{\zeta} \left\| \int_{\omega_1}^{\omega_2} x(\alpha, \beta)\, d\alpha \right\| d\beta < (\epsilon/\zeta) \cdot \zeta = \epsilon, \end{aligned}$$

the interchange of integrations being justified by (1) and (4; Theorem 3.6.7).

(ii) By (4; Theorem 3.2.3(3)),

$$y(\beta) = \int_0^{+\infty} x(\alpha, \beta)\, d\alpha$$

is a strongly measurable function of β. Thus by (4; theorem 3.5.2.), it is sufficient to show that $\|y(\beta)\|$ is in $L(0, \zeta)$. Since $y(\beta)$ is strongly measurable, $\|y(\beta)\|$ is measurable (Lebesgue). Further,

$$\begin{aligned} ||y(\beta)|| &\leqslant \int_0^{\omega} ||x(\alpha, \beta)||\, d\alpha + \left\| \int_{\omega}^{+\infty} x(\alpha, \beta)\, d\alpha \right\| \\ &\leqslant \int_0^{\omega} ||x(\alpha, \beta)||\, d\alpha + \epsilon \end{aligned}$$

for sufficiently large ω.

Thus

$$\int_0^{\zeta} ||y(\beta)||\, d\beta \leqslant \int_0^{\zeta} \int_0^{\omega} ||x(\alpha, \beta)||\, d\alpha\, d\beta + \epsilon\zeta$$

and the statement is proved.

(iii) By (i), for each $\epsilon > 0$, $\omega(\epsilon)$ exists such that for $\omega_1 > \omega(\epsilon)$

$$\left\| \int_{\omega_1}^{+\infty} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\| < \tfrac{1}{2}\epsilon.$$

Thus

$$\begin{aligned} \Gamma &= \left\| \int_0^{\zeta} \int_0^{+\infty} x(\alpha, \beta)\, d\alpha\, d\beta - \int_0^{+\infty} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\| \\ &< \left\| \int_0^{\zeta} \int_0^{+\infty} x(\alpha, \beta)\, d\alpha\, d\beta - \int_0^{\omega_1} \int_0^{\zeta} x(\alpha, \beta)\, d\beta\, d\alpha \right\| + \tfrac{1}{2}\epsilon \\ &= \left\| \int_0^{\zeta} \int_{\omega_1}^{+\infty} x(\alpha, \beta)\, d\alpha\, d\beta \right\| + \tfrac{1}{2}\epsilon \end{aligned}$$

by (1) and (4, theorem 3.6.7). But by (2), for each $\epsilon > 0$, $\omega_2(\epsilon)$ exists such that for $\omega_3 > \omega_2(\epsilon)$,

$$\left\| \int_{\omega_3}^{+\infty} x(\alpha, \beta)\, d\alpha \right\| < \epsilon.$$

Choose $\omega_1 > \max\{\omega(\epsilon), \omega_2(\epsilon/2\zeta)\}$. Then

$$\begin{aligned} \Gamma &< \left\| \int_0^{\zeta} \int_{\omega_1}^{+\infty} x(\alpha, \beta)\, d\alpha\, d\beta \right\| + \tfrac{1}{2}\epsilon \\ &\leqslant \int_0^{\zeta} \left\| \int_{\omega_1}^{+\infty} x(\alpha, \beta)\, d\alpha \right\| d\beta + \tfrac{1}{2}\epsilon = \epsilon. \end{aligned}$$

Letting $\epsilon \to 0$, the conclusion is reached.

THEOREM 2.2. *If*

(1) $x(\alpha)$ *is in* $B([0, \omega]; \mathfrak{X})$ *for each* $\omega > 0$,

(2) $\|x(\alpha)\|$ *is in* $L(0, \infty)$,

then

(i) $x(\alpha)$ *is in* $B([0, \infty); \mathfrak{X})$,

(ii) $$\int_0^{+\infty} x(\alpha)\, d\alpha \quad \textit{converges},$$

(iii) $$\int_0^{\infty} x(\alpha)\, d\alpha = \int_0^{+\infty} x(\alpha)\, d\alpha.$$

Proof.

(i) By (4; theorem 3.5.2), it is sufficient to show that $x(\alpha)$ is strongly measurable over $[0, \infty)$.

By (1), $x(\alpha)$ is strongly measurable over $[0, \omega]$ for each $\omega > 0$. Let

$$1_\omega(\alpha) = \begin{cases} 1, & 0 \leqslant \alpha \leqslant \omega, \\ 0, & \alpha > \omega. \end{cases}$$

Obviously 1_ω is Lebesgue measurable for each positive ω. Then $x_\omega(\alpha) = 1_\omega(\alpha)x(\alpha)$ is strongly measurable over $[0, \omega]$ by (4; theorem 3.2.3(2)), and since $x_\omega(\alpha) = \theta$ for $\alpha > \omega$, $x_\omega(\alpha)$ is strongly measurable over $[0, \infty)$.

Obviously

$$x(\alpha) = \lim_{\omega \to \infty} x_\omega(\alpha),$$

and thus, by (4; theorem 3.2.3(3)), $x(\alpha)$ is strongly measurable over $[0, \infty)$.

(ii) Since $\|x(\alpha)\|$ is in $L(0, \infty)$, we have,

$$\left\| \int_{\omega_1}^{\omega_2} x(\alpha)\, d\alpha \right\| \leqslant \int_{\omega_1}^{\omega_2} \|x(\alpha)\|\, d\alpha \to 0, \qquad \omega_1, \omega_2 \to \infty.$$

(iii)

$$\begin{aligned} \left\| \int_0^{+\infty} x(\alpha)\, d\alpha - \int_0^{\infty} x(\alpha)\, d\alpha \right\| &= \left\| \int_{\omega_1}^{+\infty} x(\alpha)\, d\alpha - \int_{\omega_1}^{\infty} x(\alpha)\, d\alpha \right\| \\ &\leqslant \left\| \int_{\omega_1}^{+\infty} x(\alpha)\, d\alpha \right\| + \int_{\omega_1}^{\infty} \|x(\alpha)\|\, d\alpha \to 0, \qquad \omega_1 \to \infty. \end{aligned}$$

3. Lebesgue sets. Let $x(\alpha)$ be in $B(E_1; \mathfrak{X})$ where E_1 is the one-dimensional Euclidian space. Then by (4; page 48, formula (3.6.1)),

$$\lim_{\gamma \to 0} \frac{1}{\gamma} \int_{\xi}^{\xi+\gamma} ||x(\alpha) - x(\xi)|| \, d\alpha = 0$$

almost everywhere.

DEFINITION 3.1. We shall call the set where the above formula holds the *Lebesgue set* of $x(\alpha)$.

THEOREM 3.1. *If $x(\alpha)$ is in $B(E_1; \mathfrak{X})$, and τ is in the Lebesgue set of $x(\alpha)$, then the Lebesgue set of $\|x(\alpha) - x(\tau)\|$ contains the Lebesgue set of $x(\alpha)$.*

Proof. Let ξ be in the Lebesgue set of $x(\alpha)$. Then

$$\begin{aligned} \frac{1}{\gamma} \int_{\xi}^{\xi+\gamma} & \big| \, ||x(\alpha) - x(\tau)|| - ||x(\xi) - x(\tau)|| \, \big| \, d\alpha \\ & < \frac{1}{\gamma} \int_{\xi}^{\xi+\gamma} ||(x(\alpha) - x(\tau)) - (x(\xi) - x(\tau))|| \, d\alpha \\ & = \frac{1}{\gamma} \int_{\xi}^{\xi+\gamma} ||x(\alpha) - x(\xi)|| \, d\alpha \to 0, \qquad \gamma \to 0, \end{aligned}$$

and ξ is in the Lebesgue set of $\|x(\alpha) - x(\tau)\|$.

4. A theorem on a certain singular integral. The following theorem will be needed both in the inversion and representation theories.

THEOREM 4.1. *If*
(1) *$x(\alpha)$ is in $B([0, \omega]; \mathfrak{X})$ for each $\omega > 0$,*

$$\int_0^{+\infty} e^{-\lambda\alpha} x(\alpha) \, d\alpha \tag{2}$$

converges for $\lambda = \lambda_0 > 0$, then
(i) *for each $\tau > 0$ and for all $\kappa > \lambda_0 \tau$,*

$$I_\kappa = e^{2\kappa} (\kappa/\pi\tau)^{\frac{1}{2}} \int_0^{+\infty} e^{-\kappa(\alpha\tau^{-1} + \tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha) \, d\alpha$$

converges,

$$\text{(ii)} \qquad \lim_{\kappa \to \infty} I_\kappa = x(\tau)$$

for each $\tau > 0$ in the Lebesgue set of $x(\alpha)$.

Proof.
(i) Let $\kappa > \lambda_0 \tau$, let $\omega_2 > \omega_1$, and let

$$M = \sup_{0 < \omega < \infty} \left\| \int_0^{\omega} e^{-\lambda_0 \alpha} x(\alpha) \, d\alpha \right\|.$$

By (2), $M < \infty$. Thus

$$\left\| \int_{\omega_1}^{\omega_2} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha \right\|$$

$$= \left\| e^{-\kappa(\omega_2\tau^{-1}+\tau\omega_2^{-1})+\lambda_0\omega_2} \omega_2^{-\frac{1}{2}} \int_{\omega_1}^{\omega_2} e^{-\lambda_0\beta} x(\beta)\, d\beta \right.$$

$$\left. - \int_{\omega_1}^{\omega_2} \left\{ \frac{d}{d\alpha} \left(e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})+\lambda_0\alpha} \alpha^{-\frac{1}{2}} \right) \right\} \int_{\omega_1}^{\alpha} e^{-\lambda_0\beta} x(\beta)\, d\beta\, d\alpha \right\|$$

$$< 2M \left\{ e^{-\kappa(\omega_2\tau^{-1}+\tau\omega_2^{-1})+\lambda_0\omega_2} \omega_2^{-\frac{1}{2}} + \int_{\omega_1}^{\omega_2} d\left(e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})+\lambda_0\alpha} \alpha^{-\frac{1}{2}} \right) \right\}$$

$$= 2M\, e^{-\kappa(\omega_1\tau^{-1}+\tau\omega_1^{-1})+\lambda_0\omega_1} \omega_1^{-\frac{1}{2}} \to 0, \qquad \omega_1, \omega_2 \to \infty.$$

(ii) It should be noted that

$$e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_0^{\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, d\alpha = 1.$$

Let $\tau > 0$ be in the Lebesgue set of $x(\alpha)$, and let $\omega > \tau$. Then,

$$||I_\kappa - x(\tau)|| = \left\| e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_0^{+\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} (x(\alpha) - x(\tau))\, d\alpha \right\|$$

$$< e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \left\{ \int_0^{\omega} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} ||x(\alpha) - x(\tau)||\, d\alpha \right.$$

$$\left. + \left\| \int_{\omega}^{+\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha \right\| + ||x(\tau)|| \int_{\omega}^{\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, d\alpha \right\}$$

$$= \Gamma_1 + \Gamma_2 + \Gamma_3.$$

Let $\kappa_0 > \lambda_0$ and $\kappa > \kappa_0$. Then

$$\Gamma_3 = ||x(\tau)||\, e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_{\omega}^{\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, d\alpha$$

$$= ||x(\tau)||\, e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_{\omega}^{\infty} e^{-\kappa_0(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, e^{-(\kappa-\kappa_0)(\alpha\tau^{-1}+\tau\alpha^{-1})}\, d\alpha$$

$$< ||x(\tau)||\, e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}}\, e^{-(\kappa-\kappa_0)(\omega\tau^{-1}+\tau\omega^{-1})} \int_{\omega}^{\infty} e^{-\kappa_0(\alpha\tau^{-1}+\tau\alpha^{-1})}\, d\alpha$$

$$= ||x(\tau)||(\kappa/\pi\tau)^{\frac{1}{2}}\, e^{-\kappa_0(\omega\tau^{-1}+\tau\omega^{-1})}\, e^{-\kappa(\omega\tau^{-1}+\tau\omega^{-1}-2)} \int_{\omega}^{\infty} e^{-\kappa_0(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}}\, d\alpha$$

$$\to 0, \qquad \kappa \to \infty.$$

since $\alpha\tau^{-1} + \tau\alpha^{-1} - 2$ attains its minimum value of 0 at $\alpha = \tau$, and $\omega > \tau$.

$$\Gamma_2 = \left\| e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_{\omega}^{+\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha \right\|$$

$$= \left\| e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_{\omega}^{\infty} \left\{ \frac{d}{d\alpha} \left(e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})+\lambda_0\alpha} \alpha^{-\frac{1}{2}} \right) \right\} \int_{\omega}^{\alpha} e^{-\lambda_0\beta} x(\beta)\, d\beta\, d\alpha \right\|$$

$$< 2M(\kappa/\pi\tau)^{\frac{1}{2}}\, e^{-\kappa(\omega\tau^{-1}+\tau\omega^{-1}-2)+\lambda_0\omega} \omega^{-\frac{1}{2}} \to 0, \qquad \kappa \to \infty,$$

where we have used Theorem 2.2 after the integration by parts. By Widder (**8**, p. 278, theorem 2b, corollary 2b.1), $\Gamma_1 \to 0$ as $\kappa \to \infty$ if τ is in the Lebesgue set of $\|x(\alpha) - x(\tau)\|$. However, since τ is in the Lebesgue set of $x(\alpha)$, we have, by Theorem 3.1 that τ is in the Lebesgue set of $\|x(\alpha) - x(\tau)\|$, and thus the theorem is proved.

COROLLARY. *If $e^{-\lambda_0 \alpha} x(\alpha)$ is in $B([0, \infty); \mathfrak{X})$, then*
(i) *for each $\tau > 0$ and all $\kappa > \lambda_0 \tau$,*

$$e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)$$

is in $B([0, \infty); \mathfrak{X})$,
(ii) $\lim_{\kappa\to\infty} I_\kappa = x(\tau)$ *for each $\tau > 0$ in the Lebesgue set of $x(\alpha)$ where*

$$I_\kappa = e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_0^\infty e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha.$$

Proof.
(i) This follows from the fact that for $\kappa > \lambda_0 \tau$,

$$e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} \leqslant N e^{-\lambda_0 \alpha}.$$

(ii) This now follows from Theorem 2.2 and the preceding theorem.

5. Inversion of the Laplace Transformation. We define

$$\overrightarrow{L}_{\kappa,\tau}[f(\lambda)] = (\kappa e^{2\kappa}/\pi\tau) \int_0^{+\infty} \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}}) f(\kappa(\eta + 1)/\tau)\, d\eta.$$

The following theorem provides sufficient conditions for the inversion of a Laplace transformation which involves improper Bochner integrals by

$$\overrightarrow{L}_{\kappa,\tau}[f(\lambda)].$$

THEOREM 5.1. *If $x(\alpha)$ is in $B([0, \omega]; \mathfrak{X})$ for each $\omega > 0$, and if*

$$f(\lambda) = \int_0^{+\infty} e^{-\lambda\alpha} x(\alpha)\, d\alpha$$

converges uniformly in λ for $\lambda > \gamma > 0$, then for each $\tau > 0$ and all $\kappa > \gamma\tau$, $\overrightarrow{L}_{\kappa,\tau}[f(\lambda)]$ exists, and

$$\lim_{\kappa\to\infty} \overrightarrow{L}_{\kappa,\tau}[f(\lambda)] = x(\tau)$$

at each point $\tau > 0$ of the Lebesgue set of $x(\alpha)$.

Proof. We shall show that

$$\overrightarrow{L}_{\kappa,\tau}[f(\lambda)] = e^{2\kappa}(\kappa/\pi\tau)^{\frac{1}{2}} \int_0^{+\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha,$$

and the conclusion will follow from Lemma 4.1.

Let κ and τ be fixed and $\kappa > \gamma\tau$. Choose λ_0, $\gamma < \lambda_0 < \kappa/\tau$. Let

$$M = \sup_{0<\omega<\infty} \left\| \int_0^\omega e^{-\lambda_0\alpha} x(\alpha)\, d\alpha \right\|.$$

By hypothesis, $M < \infty$. Consider

$$I_\omega = (\kappa e^{2\kappa}/\pi\tau) \int_0^\omega \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}}) f(\kappa(\eta+1)/\tau)\, d\eta$$

$$= (\kappa e^{2\kappa}/\pi\tau) \int_0^\omega \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}})\, d\eta \int_0^{+\infty} e^{-\kappa(\eta+1)\alpha/\tau} x(\alpha)\, d\alpha.$$

Since $f(\lambda)$ converges uniformly for $\lambda > \gamma$ and $\kappa > \gamma\tau$, we may, by Theorem 2.1, interchange the order of integrations. Thus

$$I_\omega = (\kappa e^{2\kappa}/\pi\tau) \int_0^{+\infty} e^{-\kappa\alpha/\tau} x(\alpha)\, d\alpha \int_0^\omega e^{-\kappa\eta\alpha/\tau} \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}})\, d\eta$$

$$= (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^{+\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha \left\{ \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right\},$$

by an obvious change of variable, where $\sigma = (\alpha\kappa\omega/\tau)^{\frac{1}{2}} + i(\kappa\tau/\alpha)^{\frac{1}{2}}$. Then

$$\Gamma_\omega = \left\| (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^{+\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha - I_\omega \right\|$$

$$\leqslant (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \left\{ \int_0^\delta e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} ||x(\alpha)|| \cdot \left| 1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right| d\alpha \right.$$

$$\left. + \left\| \int_\delta^{+\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha) \left[1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right] d\alpha \right\| \right\}$$

$$= \Gamma_\omega^{(1)} + \Gamma_\omega^{(2)} \qquad (\delta > 0).$$

Now

$$\lim_{\omega\to\infty} \left\{ 1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right\} = 0$$

almost everywhere for α in $[0, \delta]$. Thus by (4; theorem 3.2.1) the limit equals zero almost uniformly in $[0, \delta]$. Further, since

$$\int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta = \{(\kappa\tau/\alpha) + (\alpha\kappa\omega/\tau)\}^{-1} e^{(\kappa\tau/\alpha)-(\alpha\kappa\omega/\tau)} \{ -(\kappa\tau/\alpha)^{\frac{1}{2}} \sin(2\kappa\omega^{\frac{1}{2}})$$

$$- (\alpha\kappa\omega/\tau)^{\frac{1}{2}} \cos(2\kappa\omega^{\frac{1}{2}}) \} - \tfrac{1}{2} \int_{-\sigma}^{\bar{\sigma}} \theta^{-2} e^{-\theta^2}\, d\theta,$$

and

$$\left| \int_{-\sigma}^{\bar{\sigma}} \theta^{-2} e^{-\theta^2}\, d\theta \right| \leqslant M \alpha^{\frac{1}{2}} e^{\kappa\tau/\alpha},$$

where M is independent of α and ω for α in $[0, \delta]$, we have then

$$e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} ||x(\alpha)|| \cdot \left| 1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\bar{\sigma}} e^{-\theta^2}\, d\theta \right| \leqslant N e^{-\kappa\alpha/\tau},$$

and this bound is an integrable function of α. Thus by (4; theorem 3.3.6),

$$\lim_{\omega\to\infty} \Gamma_\omega^{(1)} = 0.$$

Also,

$$J_\omega^{(3)} = (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_1^{+\infty} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha) \left\{ 1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\sigma} e^{-\theta^2} d\theta \right\} d\alpha$$

converges uniformly in ω for $\omega > 0$. For, for every $\omega_2 > \omega_1 > \delta$,

$$\begin{aligned}
&\left\| \int_{\omega_1}^{\omega_2} e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha) \left\{ 1 - \pi^{-\frac{1}{2}} \int_{-\sigma}^{\sigma} e^{-\theta^2} d\theta \right\} d\alpha \right\| \\
&= \left\| 2(\kappa/\pi\tau)^{\frac{1}{2}} \int_{\omega_1}^{\omega_2} e^{-\kappa\alpha/\tau} x(\alpha)\, d\alpha \int_{\sqrt{\omega}}^{\infty} e^{-\kappa\alpha\phi^2/\tau} \cos(2\kappa\phi)\, d\phi \right\| \\
&= 2(\kappa/\pi\tau)^{\frac{1}{2}} \left\| e^{(\lambda_0-\kappa/\tau)\omega_2} \int_{\sqrt{\omega}}^{\infty} e^{-\kappa\omega_2\phi^2/\tau} \cos(2\kappa\phi)\, d\phi \int_{\omega_1}^{\omega_2} e^{-\lambda_0\alpha} x(\alpha)\, d\alpha \right. \\
&\quad + \int_{\omega_1}^{\omega_2} e^{(\lambda_0-\kappa/\tau)\alpha} \left\{ -(\lambda_0 - \kappa/\tau) \int_{\sqrt{\omega}}^{\infty} e^{-\kappa\alpha\phi^2/\tau} \cos(2\kappa\phi)\, d\phi \right. \\
&\quad \left. \left. + (\kappa/\tau) \int_{\sqrt{\omega}}^{\infty} e^{-\kappa\alpha\phi^2/\tau} \phi^2 \cos(2\kappa\phi)\, d\phi \right\} \int_{\omega_1}^{\alpha} e^{-\lambda_0\beta} x(\beta)\, d\beta\, d\alpha \right\| \\
&< 4(\kappa/\pi\tau)^{\frac{1}{2}} M \left\{ e^{(\lambda_0-\kappa/\tau)\omega_2} \int_0^{\infty} e^{-\kappa\omega_2\phi^2/\tau} d\phi \right. \\
&\quad \left. + \int_{\omega_1}^{\omega_2} e^{(\lambda_0-\kappa/\tau)\alpha} \left[-(\lambda_0 - \kappa/\tau) \int_0^{\infty} e^{-\kappa\alpha\phi^2/\tau} d\phi + (\kappa/\tau) \int_0^{\infty} e^{-\kappa\alpha\phi^2/\tau} \phi^2 d\phi \right] d\alpha \right\} \\
&= 4(\kappa/\pi\tau)^{\frac{1}{2}} M \left\{ e^{(\lambda_0-\kappa/\tau)\omega_2} \int_0^{\infty} e^{-\kappa\omega_2\phi^2/\tau} d\phi + \int_{\omega_1}^{\omega_2} d\left(e^{(\lambda_0-\kappa/\tau)\alpha} \int_0^{\infty} e^{-\kappa\alpha\phi^2/\tau} d\phi \right) \right\} \\
&= 4(\kappa/\pi\tau)^{\frac{1}{2}} M e^{(\lambda_0-\kappa/\tau)\omega_1} \int_0^{\infty} e^{-\omega_1\kappa\phi^2/\tau} d\phi \to 0, \qquad \omega_1, \omega_2 \to \infty.
\end{aligned}$$

Thus we may choose δ so large that

$$\Gamma_\omega^{(3)} = \| J_\omega^{(3)} \| < \epsilon$$

for any $\epsilon > 0$ and thus

$$\lim_{\omega \to \infty} \Gamma_\omega = 0$$

and the theorem is proved.

6. Some Lemmas. We now prove two lemmas which are preliminary to the representation theory.

LEMMA 6.1 *If*

(1) $\lambda^{-1} f(\lambda)$ *is in* $B([\delta, \infty); \mathfrak{X})$ *for each* $\delta > 0$,

$$(2) \qquad \psi(\xi) = \int_\xi^{\infty} \lambda^{-1} \| f(\lambda) \|\, d\lambda = O(\xi^{-m}),$$

with $m > \frac{1}{2}$, *as* $\xi \to \infty$, *and* $\psi(\xi) = O(e^{\gamma/\xi})$ *with* $\gamma > 0$ *as* $\xi \to 0+$, *then for each* $\kappa > 0$, *and almost all* $\tau > 0$,

$$L_{\kappa,\tau}[f(\lambda)] = (\kappa e^{2\kappa}/\pi\tau)\int_0^\infty \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\, f(\kappa(\eta+1)/\tau)\, d\eta$$

exists. In particular, $L_{\kappa,\tau}[f(\lambda)]$ exists when κ and τ are positive, and κ/τ is in the Lebesgue set of $f(\lambda)$.

Proof. The proof is the same as in (7; §3, lemma 2, page 440) if $|f(s)|$ is replaced by $\|f(\lambda)\|$ wherever it appears, and certain minor notational changes are made.

LEMMA 6.2. *If*

(1) $$\lambda^{-1}f(\lambda) \text{ is in } B([\delta,\infty);\mathfrak{X}) \qquad \text{for each } \delta>0,$$

(2) $$\psi(\xi) = \int_\xi^\infty \lambda^{-1}\|f(\lambda)\|\,d\lambda = O(\xi^{-m})$$

with $m>0$, as $\xi\to\infty$, and $\psi(\xi)=O(e^{\gamma/\xi})$ with $\gamma>0$ as $\xi\to 0+$, then for each $\epsilon>0$,

$$\int_\xi^\infty \lambda^{-1}e^{-\epsilon\lambda}\|f(\lambda)\|\,d\lambda = O(\xi^{-n}) \quad \text{for every } n>0, \text{ as } \xi\to\infty$$
$$= O(e^{\gamma/\xi}) \qquad \text{as } \xi\to 0+.$$

Proof.

$$\int_\xi^\infty \lambda^{-1}e^{-\epsilon\lambda}\|f(\lambda)\|\,d\lambda \le e^{-\epsilon\xi}\int_\xi^\infty \lambda^{-1}\|f(\lambda)\|\,d\lambda$$
$$= O(\xi^{-n}), \qquad \xi\to\infty$$
$$= O(e^{\gamma/\xi}), \qquad \xi\to 0+.$$

7. A fundamental theorem. The following theorem is fundamental in the representation theory.

THEOREM 7.1. *If*

(1) $$\lambda^{-1}f(\lambda) \text{ is in } B([\delta,\infty);\mathfrak{X}) \qquad \text{for all } \delta>0,$$

(2) $$\psi(\xi) = \int_\xi^\infty \eta^{-1}\|f(\eta)\|\,d\eta = O(\xi^{-m})$$

with $m>\frac{1}{2}$, as $\xi\to\infty$, and $\psi(\xi)=O(e^{\gamma/\xi})$, with $\gamma>0$, as $\xi\to 0+$,

(3) $e^{-\zeta\tau}L_{\kappa,\tau}[f(\lambda)]$ *is in $B([0,\infty);\mathfrak{X})$ for $\zeta>\gamma_1$, and all $\kappa>\kappa_0$,*

then

$$\lim_{\kappa\to\infty}\int_0^\infty e^{-\zeta\tau}L_{\kappa,\tau}[f(\lambda)]\,d\tau = f(\zeta)$$

at every point $\zeta>\gamma_1$, of the Lebesgue set of $f(\lambda)$.

Proof. $L_{\kappa,\tau}[f(\lambda)]$ exists by Lemma 6.1 and has a Laplace transform when $\zeta>\gamma_1$, by (3). To prove the assertion we shall use (4; theorem 3.6.7) and the corollary to Theorem 4.1.

Operating formally we have

$$\begin{aligned}
\int_0^\infty e^{-\xi\tau} L_{\kappa,\tau}[f(\lambda)]\,d\tau &= (\kappa e^{2\kappa}/\pi)\int_0^\infty e^{-\xi\tau}\tau^{-1}d\tau \int_0^\infty \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\, f(\kappa(\eta+1)/\tau)\,d\eta \\
&= (2\kappa e^{2\kappa}/\pi)\int_0^\infty e^{-\xi\tau}\tau^{-1}d\tau \int_0^\infty \cos(2\kappa\xi)\, f(\kappa(\xi^2+1)/\tau)\,d\xi \\
&= (2\kappa e^{2\kappa}/\pi)\int_0^\infty \cos(2\kappa\xi)\,d\xi \int_0^\infty e^{-\xi\tau}\,\tau^{-1} f(\kappa(\xi^2+1)/\tau)\,d\xi \\
&= (2\kappa e^{2\kappa}/\pi)\int_0^\infty \cos(2\kappa\xi)\,d\xi \int_0^\infty e^{-\kappa\xi\beta(\xi^2+1)}\,\beta^{-1} f(\beta^{-1})\,d\beta \\
&= (2\kappa e^{2\kappa}/\pi)\int_0^\infty e^{-\kappa\xi\beta}\,\beta^{-1} f(\beta^{-1})\,d\beta \int_0^\infty e^{-\kappa\xi\beta\xi^2}\cos(2\kappa\xi)\,d\xi \\
&= (\kappa/\pi\xi)^{\frac{1}{2}}\, e^{2\kappa}\int_0^\infty e^{-\kappa(\xi\beta+(\xi\beta)^{-1})}\,\beta^{-3/2} f(\beta^{-1})\,d\beta \\
&\to f(\xi), \qquad \kappa\to\infty.
\end{aligned}$$

These formal calculations will be justified if the two interchanges of integrations are justified, and the conditions of Lemma 4.1 are met.

For the first interchange of integrations it is sufficient to show that

$$\int_0^\infty |\cos(2\kappa\xi)|\,d\xi \int_0^\infty e^{-\kappa\xi\beta(\xi^2+1)}\,\beta^{-1}\,\|f(\beta^{-1})\|\,d\beta < \infty.$$

But by (2) and (7; §3, lemma 1, page 438), if $\kappa\xi > \gamma$, the inner integral is $O(\xi^{-2m})$ as $\xi\to\infty$, and $m > \frac{1}{2}$.

For the second interchange it is sufficient to show that

$$\int_0^\infty e^{-\kappa\xi\beta}\,\beta^{-1}\,\|f(\beta^{-1})\|\,d\beta \int_0^\infty |e^{-\kappa\xi\beta\xi^2}\cos(2\kappa\xi)|\,d\xi < \infty.$$

But this is so since the inner integral is less than $\frac{1}{2}(\pi/(\kappa\xi\beta))^{\frac{1}{2}}$ and

$$\int_0^\infty e^{-\kappa\xi\beta}\,\beta^{-3/2}\,\|f(\beta^{-1})\|\,d\beta < \infty$$

by (1), (2), and (7; §3, lemma 1, page 438).

8. Representation of abstractly-valued functions by Laplace transformations of functions in $B_p([0,\infty);\mathfrak{X})$, $1 < p < \infty$. In this section we find necessary and sufficient conditions that a function $f(\lambda)$ on $(0,\infty)$ to a Banach space $\mathfrak{X}$ be represented as the Laplace integral of a function in $B_p([0,\infty);\mathfrak{X})$ where $1 < p < \infty$.

In order to obtain such conditions for these general classes of functions, we find it necessary to postulate some sort of compactness condition on $B_p([0,\infty);\mathfrak{X})$. We have chosen the weakest condition possible, namely weak compactness of the unit sphere in $B_p([0,\infty);\mathfrak{X})$. By Bochner and Taylor (1), Pettis (6), and Eberlein (3), a necessary and sufficient condition for this compactness is that $\mathfrak{X}$ be reflexive.

There is one other complication that must be dealt with. That is that a function $f(\lambda)$ may be the Laplace transform of a function $x(\alpha)$ in $B_p([0, \infty); \mathfrak{X})$, $1 < p \leqslant 2$, and yet $L_{\kappa,\tau}[f(\lambda)]$ may not exist. For example, let $f(\lambda) = (\lambda + 1)^{-1/2}$. To overcome this difficulty we resort to what is essentially Cauchy's method of summation.

We define

$$L^*_{\kappa,\tau}[f(\lambda)] = L_{\kappa,\tau}[e^{-\epsilon\lambda} f(\lambda)].$$

The following theorem gives the necessary and sufficient conditions in question.

THEOREM 8. *If $\mathfrak{X}$ is a reflexive Banach space, then the following conditions are necessary and sufficient for $f(\lambda)$ to be equal almost everywhere for $\lambda > 0$ to the Laplace integral of a function in $B_p([0, \infty); \mathfrak{X})$, p fixed, $1 < p < \infty$.*

(1) $\lambda^{-1} f(\lambda)$ *is in* $B([\delta, \infty); \mathfrak{X})$ *for all* $\delta > 0$,

(2) $$\int_\xi^\infty \lambda^{-1} \| f(\lambda) \| \, d\lambda = O(\xi^{-m}) \quad \text{with } m > 0 \text{ as } \xi \to \infty$$
$$= O(e^{\gamma/\xi}) \quad \text{with } \gamma > 0 \text{ as } \xi \to 0+,$$

(3) $\|L^*_{\kappa,\tau}[f(\lambda)]\|_p < M_p$, *where M_p is independent of ϵ and κ, p fixed, $\kappa > \kappa_0$, $\epsilon > 0$.*

Proof of necessity. Suppose

$$f(\lambda) = \int_0^\infty e^{-\lambda\alpha} x(\alpha) \, d\alpha$$

a.e. for $\lambda > 0$, and $x(\alpha)$ is in $B_p([0, \infty); \mathfrak{X})$. Then using Hölder's inequality we have

$$\lambda^{-1}\|f(\lambda)\| \leqslant \lambda^{-1} \int_0^\infty e^{-\lambda\alpha} \|x(\alpha)\| \, d\alpha \leqslant \lambda^{-1} \left\{ \int_0^\infty e^{-q\lambda\alpha} d\alpha \right\}^{1/q} \left\{ \int_0^\infty \|x(\alpha)\|^p \, d\alpha \right\}^{1/p}$$
$$= A\lambda^{-1-1/q},$$

so that (1) is necessary. Thus

$$\int_\xi^\infty \lambda^{-1} \|f(\lambda)\| \, d\lambda \leqslant B\xi^{-1/q},$$

so that (2) is necessary. Now

$$L^*_{\kappa,\tau}[f(\lambda)] = L_{\kappa,\tau}[e^{-\epsilon\lambda} f(\lambda)]$$
$$= (2\kappa e^{2\kappa}/\pi\tau) \int_0^\infty \cos(2\kappa\eta) \, e^{-\kappa(\eta^2+1)/\tau} \, d\eta \int_0^\infty e^{-\kappa(\eta^2+1)\alpha/\tau} x(\alpha) \, d\alpha$$
$$= (2\kappa e^{2\kappa}/\pi\tau) \int_0^\infty e^{-\kappa(\alpha+\epsilon)/\tau} x(\alpha) \, d\alpha \int_0^\infty e^{-\kappa(\alpha+\epsilon)\eta^2/\tau} \cos(2\kappa\eta) \, d\eta$$
$$= (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha + \epsilon)^{-\frac{1}{2}} x(\alpha) \, d\alpha,$$

the interchange of integration being justified by the fact that

$$\int_0^\infty e^{-\kappa(\alpha+\epsilon)/\tau} \,||\, x(\alpha) \,||\, d\alpha \int_0^\infty e^{-\kappa(\alpha+\epsilon)\eta^2/\tau} \,|\cos(2\kappa\eta)|\, d\eta$$
$$< B \int_0^\infty e^{-\kappa\alpha/\tau} \,||\, x(\alpha) \,||\, d\alpha.$$

Thus

$$\begin{aligned}
||\, L_{\kappa,\tau}^{\epsilon}[f(\lambda)] \,|| &< (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha+\epsilon)^{-\frac{1}{2}} \,||\, x(\alpha) \,||\, d\alpha \\
&< (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \left\{ \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha+\epsilon)^{-\frac{1}{2}} \,||\, x(\alpha) ||^p d\alpha \right\}^{1/p} \\
&\quad \cdot \left\{ \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha+\epsilon)^{-\frac{1}{2}} d\alpha \right\}^{1/q} \\
&< (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \left\{ \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha+\epsilon)^{-\frac{1}{2}} \,||\, x(\alpha) ||^p d\alpha \right\}^{1/p} \\
&\quad \cdot \left\{ \int_0^\infty e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} d\alpha \right\}^{1/q} \\
&= (\kappa/\pi\tau)^{\frac{1}{2}p} e^{2\kappa/p} \left\{ \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\alpha+\epsilon)^{-\frac{1}{2}} \,||\, x(\alpha) \,||^p d\alpha \right\}^{1/p}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&||\, L_{\kappa,\cdot}^{\epsilon}[f(\lambda)] \,||_p \\
&\quad < \left\{ (\kappa/\pi)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty \int_0^\infty e^{-\kappa((\alpha+\epsilon)\tau^{-1}+\tau(\alpha+\epsilon)^{-1})} (\tau(\alpha+\epsilon))^{-\frac{1}{2}} \,||\, x(\alpha) \,||^p \, d\tau \, d\alpha \right\}^{1/p} \\
&\quad = \left\{ \int_0^\infty ||\, x(\alpha) \,||^p \, d\alpha \right\}^{1/p} = ||\, x(\cdot) \,||_p
\end{aligned}$$

and (3) is necessary.

Proof of sufficiency. By (1), (2), (3), lemma 6.2, and theorem 7.1,

$$e^{-\epsilon\zeta} f(\zeta) = \lim_{\kappa\to\infty} \int_0^\infty e^{-\zeta\tau} L_{\kappa,\tau}^{\epsilon}[f(\lambda)]\, d\tau$$

a.e. for $\zeta > 0$.

By **(1)** and **(6)**, if $\mathfrak{X}$ is a reflexive Banach space, $B_p([0,\infty);\mathfrak{X})$, $1 < p < \infty$ is reflexive, and by **(3)** a reflexive Banach space has a weakly compact unit sphere. Thus $B_p([0,\infty);\mathfrak{X})$ has a weakly compact unit sphere, and thus, since

$$||\, L_{\kappa,\cdot}^{\epsilon}[f(\lambda)] \,||_p < M_p,$$

for each $\epsilon > 0$ there exists an element $x_\epsilon(\cdot)$ of $B_p([0,\infty);\mathfrak{X})$ and an increasing unbounded sequence $\{{}_\epsilon\kappa_t\}$ such that for every y^* in $B_p^*([0,\infty);\mathfrak{X})$,

$$\lim_{t\to\infty} y^*(L_{{}_\epsilon\kappa_t,\cdot}^{\epsilon}[f(\lambda)]) = y^*(x_\epsilon(\cdot)).$$

Further, since

$$|\, y^*(L_{{}_\epsilon\kappa_t,\cdot}^{\epsilon}[f(\lambda)]) \,| < ||\, y^* \,||\cdot ||\, L_{{}_\epsilon\kappa_t,\cdot}^{\epsilon}[f(\lambda)] \,||_p$$

we have, by **(4**; theorem 2.12.3**)**, $\|x_\epsilon(\cdot)\|_p < M_p$.

Let x^* be an arbitrary element of $\mathfrak{X}^*$. Then if $g(\cdot)$ is an arbitrary element of $B_p([0, \infty); \mathfrak{X})$,

$$x^*\left(\int_0^\infty e^{-\zeta\alpha} g(\alpha)\, d\alpha\right) = \int_0^\infty e^{-\zeta\alpha} x^*(g(\alpha))\, d\alpha = y_\zeta^*(g(\cdot))$$

defines an element

$$y_\zeta^* \text{ of } B_p^*([0, \infty); \mathfrak{X})$$

for each $\zeta > 0$. For, y_ζ^* is obviously linear, and using Hölder's inequality we have,

$$|y_\zeta^*(g(\cdot))| = \left|\int_0^\infty e^{-\zeta\alpha} x^*(g(\alpha))\, d\alpha\right| < \left\{\int_0^\infty e^{-q\zeta\alpha} d\alpha\right\}^{1/q}\left\{\int_0^\infty |x^*(g(\alpha))|^p d\alpha\right\}^{1/p}$$

$$< (q\zeta)^{-1/q} \|x^*\| \left\{\int_0^\infty \|g(\alpha)\|^p\, d\alpha\right\}^{1/p} = (q\zeta)^{-1/q} \|x^*\| \, \|g(\cdot)\|_p$$

so that y_ζ^* is bounded for each $\zeta > 0$.

Thus we have, for almost every $\zeta > 0$, each $\epsilon > 0$, and every x^* in $\mathfrak{X}^*$,

$$x^*(e^{-\epsilon\zeta} f(\zeta)) = x^*\left(\lim_{t\to\infty} \int_0^\infty e^{-\zeta\tau} L_\epsilon^\epsilon \kappa_{t,\tau}\, [f(\lambda)]\, d\tau\right),$$

$$\lim_{t\to\infty} y_\zeta^*\,(L_\epsilon^\epsilon \kappa_{t,} \cdot [f(\lambda)]) = y_\zeta^*\,(x_\epsilon(\cdot)) = x^*\left(\int_0^\infty e^{-\zeta\tau} x_\epsilon(\tau)\, d\tau\right).$$

Thus for each $\epsilon > 0$, and all $\zeta > 0$ not in a set Σ_ϵ of measure zero,

$$e^{-\epsilon\zeta} f(\zeta) = \int_0^\infty e^{-\zeta\tau} x_\epsilon(\tau)\, d\tau.$$

Now since $\|x_\epsilon(\cdot)\|_p \leqslant M_p$ for each $\epsilon > 0$, and since $B_p([0, \infty); \mathfrak{X})$ has a weakly compact unit sphere, there exists an element $x(\cdot)$ of $B_p([0, \infty); \mathfrak{X})$ and a sequence $\{\epsilon_i\}$ with

$$\lim_{i\to\infty} \epsilon_i = 0$$

such that for every y^* in $B_p^*([0, \infty); \mathfrak{X})$,

$$\lim_{i\to\infty} y^*(x_{\epsilon_i}(\cdot)) = y^*(x(\cdot)).$$

Let

$$\Sigma = \bigcup_i \Sigma_{\epsilon_i}.$$

Then Σ has measure zero. Further, let ζ be in $(0, \infty) - \Sigma$. Then for each x^* in $\mathfrak{X}^*$

$$x^*(f(\zeta)) = \lim_{i\to\infty} x^*(e^{-\epsilon_i\zeta} f(\zeta)) = \lim_{i\to\infty} x^*\left(\int_0^\infty e^{-\zeta\tau} x_{\epsilon_i}(\tau)\, d\tau\right)$$

$$= \lim_{i\to\infty} y_\zeta^*\,(x_{\epsilon_i}(\cdot)) = y_\zeta^*\,(x(\cdot)) = x^*\left(\int_0^\infty e^{-\zeta\tau} x(\tau)\, d\tau\right).$$

Thus

$$f(\zeta) = \int_0^\infty e^{-\zeta\tau} x(\tau)\, d\tau$$

a.e. for $\zeta > 0$, and $x(\cdot)$ is in $B_p([0, \infty); \mathfrak{X})$.

9. Representation of Abstractly-valued functions by Laplace Transformations of functions in $B_\infty([0, \infty); \mathfrak{X})$. The following two theorems give necessary and sufficient conditions that a function on $(0, \infty)$ to a uniformly convex Banach space $\mathfrak{X}$ be equal almost everywhere to a Laplace integral of a function in $B_\infty([0, \infty); \mathfrak{X})$.

It will be noted that the representation theorem is stated in terms of $L_{s,\tau}[f(\lambda)]$. The theorem could equally well be stated in terms of $L^{\epsilon}_{s,\tau}[f(\lambda)]$, but we have chosen this form for simplicity.

THEOREM 9.1. *If $\{T_\sigma\}$, $0 < \sigma < \infty$, is a set of linear transformations on a separable Banach space $\mathfrak{X}$ to a reflexive Banach space $\mathfrak{Y}$, and if $\|T_\sigma\| \leqslant M$ independent of σ, for all $\sigma > 0$, then there exists an increasing unbounded sequence $\{\sigma_t\}$ and a linear transformation T on $\mathfrak{X}$ to $\mathfrak{Y}$ with $\|T\| \leqslant M$, such that*

$$\lim_{t\to\infty} y^*(T_{\sigma_t}(x)) = y^*(T(x))$$

for every x in $\mathfrak{X}$ and every y^* in $\mathfrak{Y}^*$.

Proof. Let $D = \{x_n\}$ be a countable set dense in $\mathfrak{X}$. Since $\mathfrak{Y}$ is reflexive, it has, by **(3)**, a weakly compact unit sphere, so that there exists an increasing unbounded sequence $\sigma_{t,1}$ and an element y_1 of $\mathfrak{Y}$ such that for every y^* in $\mathfrak{Y}^*$,

$$\lim_{t\to\infty} y^*(T_{\sigma_{t,1}}(x_1)) = y^*(y_1).$$

Further, there exists an increasing unbounded sequence $\{\sigma_{t,2}\} \subseteq \{\sigma_{t,1}\}$ and an element y_2 of $\mathfrak{Y}$ such that for every y^* in $\mathfrak{Y}^*$,

$$\lim_{t\to\infty} y^*(T_{\sigma_{t,2}}(x_2)) = y^*(y_2).$$

Inductively, there exists an increasing unbounded sequence $\{\sigma_{t,n}\} \subseteq \{\sigma_{t,n-1}\}$ and an element y_n of $\mathfrak{Y}$ such that for every y^* in $\mathfrak{Y}^*$

$$\lim_{t\to\infty} y^*(T_{\sigma_{t,n}}(x_n)) = y^*(y_n).$$

Thus, using the diagonal sequence, we have for every y^* in $\mathfrak{Y}^*$,

$$\lim_{t\to\infty} y^*(T_{\sigma_{t,t}}(x_j)) = y^*(y_j).$$

Further

$$|y^*(y_j)| = \lim_{t\to\infty} |y^*(T_{\sigma_{t,t}}(x_j))| \leqslant ||y^*|| \cdot M \cdot ||x_j||,$$

so that, by **(4**; theorem 2.12.3**)**, $\|y_n\| \leqslant M\|x_n\|$.

We define $\sigma_t = \sigma_{t,t}$, $T(x_n) = y_n$, $T(\alpha x_m + \beta x_n) = \alpha y_m + \beta y_n$. The last part of the definition of T is consistent with the first part, since, if $x_l = \alpha x_m + \beta x_n$, then for every y^* in $\mathfrak{Y}^*$,

$$\begin{aligned} y^*(y_l) &= \lim_{t\to\infty} y^*(T_{\sigma_t}(x_l)) = \lim_{t\to\infty} y^*(T_{\sigma_t}(\alpha x_m + \beta x_n)) \\ &= \lim_{t\to\infty} \{\alpha y^*(T_{\sigma_t}(x_m)) + \beta y^*(T_{\sigma_t}(x_n))\} = \alpha y^*(y_m) + \beta y^*(y_n) \\ &= y^*(\alpha y_m + \beta y_n), \end{aligned}$$

so that $y_l = \alpha y_m + \beta y_n$, or $T(x_l) = \alpha T(x_m) + \beta T(x_n)$. Obviously we have on this linear manifold, $\|T\| \leqslant M$.

Now let x be an arbitrary element of $\mathfrak{X}$. Then there is a sequence $\{x_{n_i}\} \subseteq D$ such that

$$\lim_{j\to\infty} x_{n_j} = x.$$

Further, if $y_{n_i} = T(x_{n_i})$,

$$\lim_{j\to\infty} y_{n_j}$$

exists; for,

$$\| y_{n_j} - y_{n_k} \| = \| T(x_{n_j} - x_{n_k}) \| \leqslant M \| x_{n_j} - x_{n_k} \| \to 0, \qquad j, k, \to \infty.$$

Further if $\bar{x}_{n_i}$ is any other sequence whose limit is x, and if $\bar{y}_{n_i} = T(\bar{x}_{n_i})$, then

$$\| y_{n_j} - \bar{y}_{n_l} \| = \| T(x_{n_j} - \bar{x}_{n_l}) \| \leqslant M \| x_{n_j} - \bar{x}_{n_l} \| \to 0, \qquad j, l \to \infty$$

since the sequences have the same limit. We define

$$T(x) = \lim y_{n_i}.$$

It is evident that T is bounded and linear on $\mathfrak{X}$, and, in fact $\|T\| \leqslant M$.

Also we have

$$\lim_{i\to\infty} y^*(T_{\sigma_i}(x)) = y^*(T(x))$$

for every y^* in $\mathfrak{Y}^*$. For,

$$\begin{aligned}
&| y^*(T_{\sigma_i}(x)) - y^*(T(x))| \\
&= | y^*(T_{\sigma_i}(x - x_{n_j})) + (y^*(T_{\sigma_i}(x_{n_j})) - y^*(T(x_{n_j}))) + y^*(T(x - x_{n_j})) | \\
&\leqslant 2 \| y^* \| \cdot M \cdot \| x - x_{n_j} \| + | y^*(T_{\sigma_i}(x_{n_j})) - y^*(T(x_{n_j})) | \to 0, \quad i, j \to \infty.
\end{aligned}$$

Thus, the theorem is proved.

THEOREM 9.2. *If $\mathfrak{X}$ is a uniformly convex Banach space, then the following conditions are necessary and sufficient that $f(\lambda)$ be equal almost everywhere, for $\lambda > 0$, to a Laplace integral of a function in $B_\infty([0, \infty); \mathfrak{X})$.*

(1) $\qquad \lambda^{-1} f(\lambda)$ *is in* $B_\infty([\delta, \infty); \mathfrak{X})$ $\qquad$ *for all* $\delta > 0$,

(2) $$\int_\xi^\infty \lambda^{-1} \| f(\lambda) \| \, d\lambda = O(\xi^{-m}) \qquad \text{with } m > \tfrac{1}{2}, \qquad \xi \to \infty$$
$$= O(e^{\gamma/\xi}) \qquad \text{with } \gamma > 0, \qquad \xi \to 0+,$$

(3) $$\| L_\kappa \cdot [f(\lambda)] \|_\infty \leqslant M_\infty, \qquad \kappa > \kappa_0.$$

Proof of necessity. Suppose

$$f(\lambda) = \int_0^\infty e^{-\lambda\tau} x(\tau) \, d\tau,$$

where $x(\tau)$ is in $B_\infty([0, \infty); \mathfrak{X})$. Then for $\lambda > 0$,

$$\lambda^{-1} \| f(\lambda) \| \leqslant \lambda^{-1} \int_0^\infty e^{-\lambda\tau} \| x(\tau) \| \, d\tau \leqslant \lambda^{-1} \| x(\cdot) \|_\infty \int_0^\infty e^{-\lambda\tau} \, d\tau = \lambda^{-2} \| x(\cdot) \|_\infty,$$

so that (1) is necessary. Further then,

$$\int_\xi^\infty \lambda^{-1} \| f(\lambda) \| \, d\lambda < \xi^{-1} \| x(\cdot) \|_\infty,$$

so that (2) is necessary. Finally, we have

$$\begin{aligned} L_{\kappa,\tau}[f(\lambda)] &= (2\kappa e^{2\kappa}/(\pi\tau)) \int_0^\infty \cos(2\kappa\eta)\, d\eta \int_0^\infty e^{-\kappa(\eta^2+1)\alpha/\tau} x(\alpha)\, d\alpha \\ &= (2\kappa e^{2\kappa}/\pi\tau) \int_0^\infty e^{-\kappa\alpha/\tau} x(\alpha)\, d\alpha \int_0^\infty e^{-\kappa\eta^2\alpha/\tau} \cos(2\kappa\eta)\, d\eta \\ &= (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha, \end{aligned}$$

the interchange of integrations being justified by the fact that

$$\begin{aligned} \int_0^\infty |\cos(2\kappa\eta)|\, d\eta \int_0^\infty e^{-\kappa(\eta^2+1)\alpha/\tau} \| x(\alpha) \|\, d\alpha \\ < (\tau \| x(\cdot) \|_\infty/\kappa) \int_0^\infty (\eta^2+1)^{-1} d\eta = \pi\tau \| x(\cdot) \|_\infty/2\kappa < \infty. \end{aligned}$$

Thus

$$\begin{aligned} \| L_{\kappa,\tau}[f(\lambda)] \| &< (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} \| x(\alpha) \|\, d\alpha \\ &< \| x(\cdot) \|_\infty (\kappa/\pi\tau)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty e^{-\kappa(\alpha\tau^{-1}+\tau\alpha^{-1})} \alpha^{-\frac{1}{2}} d\alpha \\ &= \| x(\cdot) \|_\infty. \end{aligned}$$

Hence

$$\| L_{\kappa,\cdot}[f(\lambda)] \|_\infty = \sup_{0<\tau<\infty} \| L_{\kappa,\tau}[f(\lambda)] \| < \| x(\cdot) \|_\infty.$$

Proof of sufficiency. By (1), (2), (3) and Theorem 7.1, we have, for almost all $\xi > 0$,

$$f(\xi) = \lim_{\kappa\to\infty} \int_0^\infty e^{-\xi\tau} L_{\kappa,\tau}[f(\lambda)]\, d\tau.$$

By **(5)**, a uniformly convex Banach space is reflexive, so that $\mathfrak{X}$ is reflexive. Let ϕ be in $L_1(0, \infty)$. Define

$$T_\kappa(\phi) = \int_0^\infty \phi(\tau) L_{\kappa,\tau}[f(\lambda)]\, d\tau.$$

Then

$$\begin{aligned} \| T_\kappa(\phi) \| &< \int_0^\infty |\phi(\tau)| \cdot \| L_{\kappa,\tau}[f(\lambda)] \|\, d\tau \\ &< \| L_{\kappa,\cdot}[f(\lambda)] \|_\infty \int_0^\infty |\phi(\tau)|\, d\tau < M_\infty \| \phi \|_1. \end{aligned}$$

Thus $T_\kappa(\phi)$ is a set of linear transformations on a separable space, $L_1(0, \infty)$, to a reflexive space $\mathfrak{X}$, and so, by Theorem 9.1, there is an increasing unbounded sequence $\{\kappa_i\}$, and a linear transformation T on L_1 to $\mathfrak{X}$, such that for every x^* in $\mathfrak{X}^*$, and every ϕ in L_1,

$$\lim_{t\to\infty} x^*(T_{s,t}(\phi)) = x^*(T(\phi)).$$

But by **(2)**, every bounded linear transformation T on L_1 to a uniformly convex Banach space $\mathfrak{X}$ is of the form

$$T(\phi) = \int_0^\infty \phi(\tau)\, x(\tau)\, d\tau,$$

where $x(\tau)$ is in $B_\infty([0, \infty); \mathfrak{X})$. Thus $x(\tau)$ in $B_\infty([0, \infty); \mathfrak{X})$ exists so that T has the above form, and then we must have for every x^* in $\mathfrak{X}^*$,

$$\lim_{t\to\infty} x^*\left(\int_0^\infty \phi(\tau)\, L_{s,t,\tau}[f(\lambda)]\, d\tau\right) = \lim_{t\to\infty} x^*(T_{s,t}(\phi))$$
$$= x^*(T(\phi)) = x^*\left(\int_0^\infty \phi(\tau)\, x(\tau)\, d\tau\right).$$

Let $\phi(\tau) = e^{-\zeta\tau}$. Then, for almost all $\zeta > 0$,

$$x^*(f(\zeta)) = \lim_{t\to\infty} x^*\left(\int_0^\infty e^{-\zeta\tau} L_{s,t,\tau}[f(\lambda)]\, d\tau\right)$$
$$= \lim_{t\to\infty} x^*(T_{s,t}(e^{-\zeta\tau})) = x^*(T(e^{-\zeta\tau}))$$
$$= x^*\left(\int_0^\infty e^{-\zeta\tau} x(\tau)\, d\tau\right).$$

so that for almost all $\zeta > 0$,

$$f(\zeta) = \int_0^\infty e^{-\zeta\tau} x(\tau)\, d\tau.$$

10. Representation Theorems for $f(\lambda)$ in $H_p(\alpha; \mathfrak{X})$. The class $H_p(\alpha; \mathfrak{X})$ is defined in **(4**; definition 10.4**)**. The following two theorems give the conditions under which a function in $H_p(\alpha; \mathfrak{X})$ can be represented as a Laplace integral.

THEOREM 10.1. *If $f(\lambda)$ is in $H_1(\alpha; \mathfrak{X})$ where $\alpha > 0$, then*

$$\lim_{s\to\infty} L_{s,\tau}[f(\lambda)]$$

exists and equals

$$g(\tau) = (2\pi i)^{-1} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\tau\mu} f(\mu)\, d\mu$$

and

$$f(\lambda) = \int_0^\infty e^{-\lambda\tau} g(\tau)\, d\tau.$$

Proof. By **(4**; theorem 10.4.1**)** we have, for $\Re\lambda > \alpha$,

$$f(\lambda) = (2\pi i)^{-1} \int_{\alpha-i\infty}^{\alpha+i\infty} f(\mu)(\lambda - \mu)^{-1}\, d\mu.$$

Thus

$$L_{\kappa,\tau}[f(\lambda)] = (2\kappa e^{2\kappa}/\pi\tau)\int_0^\infty \cos(2\kappa\eta)\, f(\kappa(\eta^2+1)/\tau)\, d\eta$$
$$= (\kappa e^{2\kappa}/\pi^2\tau)\int_0^\infty \cos(2\kappa\xi)d\xi \int_{-\infty}^\infty f(\alpha+i\eta)\{(\kappa(\xi^2+1)/\tau)-(\alpha+i\eta)\}^{-1}d\eta$$
$$= (e^{2\kappa}/\pi^2)\int_{-\infty}^\infty f(\alpha+i\eta)\, d\eta \int_0^\infty \cos(2\kappa\xi)\{\xi^2+(1-\tau(\alpha+i\eta)/\kappa)\}^{-1}d\xi$$

the interchange of integrations being valid for $\kappa > \alpha\tau$ since

$$\int_{-\infty}^\infty ||f(\alpha+i\eta)||\, d\eta \int_0^\infty |\cos(2\kappa\xi)\{\xi^2+(1-\tau(\alpha+i\eta)/\kappa)\}^{-1}|\, d\xi$$
$$\leqslant \int_{-\infty}^\infty ||f(\alpha+i\eta)||\, d\eta \int_0^\infty (\xi^2+(1-\tau\alpha/\kappa))^{-1}\, d\xi$$
$$\leqslant \tfrac{1}{2}\pi(1-\tau\alpha/\kappa)^{-\frac{1}{2}}\,||f||_1 < \infty.$$

Thus if $\kappa > \alpha\tau$,

$$L_{\kappa,\tau}[f(\lambda)] = (e^{2\kappa}/\pi)\int_{-\infty}^\infty e^{-2\kappa(1-\tau(\alpha+i\eta)/\kappa)^{1/2}}\,(1-\tau(\alpha+i\eta)/\kappa)^{-\frac{1}{2}} f(\alpha+i\eta)\, d\eta.$$

It is evident that for each η and each $\tau > 0$,

$$\lim_{\kappa\to\infty} e^{2\kappa(1-(1-\tau(\alpha+i\eta)/\kappa)^{1/2})}\,(1-\tau(\alpha+i\eta)/\kappa)^{-\frac{1}{2}} = e^{\tau(\alpha+i\eta)}.$$

Further, a lengthy but straightforward calculation shows that the maximum value of

$$|\, e^{2\kappa(1-(1-\tau(\alpha+i\eta)/\kappa)^{1/2})}\,(1-\tau(\alpha+i\eta)/\kappa)^{-\frac{1}{2}} - e^{\tau(\alpha+i\eta)}|$$

occurs at $\eta = 0$. Thus

$$||\, L_{\kappa,\tau}[f(\lambda)] - g(\tau)\,||$$
$$= ||\,(2\pi)^{-1}\int_{-\infty}^\infty (e^{2\kappa(1-(1-\tau(\alpha+i\eta)/\kappa)^{1/2})}(1-\tau(\alpha+i\eta)/\kappa)^{-\frac{1}{2}} - e^{\tau(\alpha+i\eta)})\, f(\alpha+i\eta)\, d\eta\,||$$
$$\leqslant (2\pi)^{-1}\,|\, e^{2\kappa(1-(1-\tau\alpha/\kappa)^{1/2})}\,(1-\tau\alpha/\kappa)^{-\frac{1}{2}} - e^{\tau\alpha}\,|\int_{-\infty}^\infty ||f(\alpha+i\eta)||\, d\eta \to 0, \quad \kappa\to\infty.$$

Thus

$$\lim_{\kappa\to\infty} L_{\kappa,\tau}[f(\lambda)] = (2\pi i)^{-1}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\tau\mu} f(\mu)\, d\mu = g(\tau).$$

Hence we have, for $\Re\lambda > \alpha$

$$\int_0^\infty e^{-\lambda\tau}\, g(\tau)\, d\tau = (2\pi)^{-1}\int_0^\infty e^{-\lambda\tau}\, d\tau \int_{-\infty}^\infty e^{\tau(\alpha+i\eta)} f(\alpha+i\eta)\, d\eta$$
$$= (2\pi)^{-1}\int_{-\infty}^\infty f(\alpha+i\eta)\, d\eta \int_0^\infty e^{-(\lambda-(\alpha+i\eta))\tau}\, d\tau$$
$$= (2\pi)^{-1}\int_{-\infty}^\infty f(\alpha+i\eta)(\lambda-(\alpha+i\eta))^{-1}\, d\eta$$
$$= (2\pi i)^{-1}\int_{\alpha-i\infty}^{\alpha+i\infty} f(\mu)(\lambda-\mu)^{-1}\, d\mu = f(\lambda).$$

The interchange of integrations is valid for $\Re\lambda > \alpha$ since

$$\int_{-\infty}^{\infty} || e^{\tau(\alpha+i\eta)} f(\alpha + i\eta) || \, d\eta = e^{\alpha\tau} \int_{-\infty}^{\infty} || f(\alpha + i\eta) || \, d\eta.$$

THEOREM 10.2. *If*

(1) $$f(\lambda) \text{ is in } H_p(\alpha; \mathfrak{X}),\ p > 1, \alpha > 0,$$

(2) $$p^{-1} + q^{-1} = 1,$$

(3) $$\beta q > 1,$$

then

$$f(\lambda) = \lambda^{\beta} \int_0^{\infty} e^{-\lambda\tau} g_\beta(\tau) \, d\tau,$$

where

$$g_\beta(\tau) = \lim_{\kappa\to\infty} L_{\kappa,\tau} [\lambda^{-\beta} f(\lambda)] = (2\pi i)^{-1} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\tau\mu} \mu^{-\beta} f(\mu) \, d\mu.$$

Proof. $\lambda^{-\beta} f(\lambda)$ is in $H_1(\alpha; \mathfrak{X})$; for using Hölder's inequality we have

$$\int_{-\infty}^{\infty} || (\alpha + i\eta)^{-\beta} f(\alpha + i\eta) || \, d\eta$$

$$< \left\{ \int_{-\infty}^{\infty} || f(\alpha + i\eta) ||^p \, d\eta \right\}^{1/p} \left\{ \int_{\infty}^{-\infty} | \alpha + i\eta) |^{-\beta q} \, d\eta \right\}^{1/q} < \infty.$$

Thus applying the previous theorem to $\lambda^{-\beta} f(\lambda)$, we have

$$\lim_{\kappa\to\infty} L_{\kappa,\tau} [\lambda^{-\beta} f(\lambda)] = (2\pi i)^{-1} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\tau\mu} \mu^{-\beta} f(\mu) \, d\mu = g_\beta(\tau),$$

$$f(\lambda) = \lambda^{\beta} \int_0^{\infty} e^{-\lambda\tau} g_\beta(\tau) \, d\tau.$$

REFERENCES

1. S. Bochner and A. E. Taylor, *Linear functionals on certain spaces of abstractly monotone functions*, Ann. Math., *39* (1938), 913–944.
2. N. Dunford, *Integration and linear operations*, Trans. Amer. Math. Soc., *40* (1936), 474–494.
3. W. F. Eberlein, *Weak compactness in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A., *33* (1947), 51–53.
4. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publications, *31* (New York, 1948).
5. B. J. Pettis, *A proof that every uniformly convex space is reflexive*, Duke Math. J., *5* (1939), 249–253.
6. ———, *Differentiation in Banach spaces*, Duke Math. J., *5* (1939), 254–269.
7. P. G. Rooney, *A new inversion and representation theory for the Laplace transformation*, Can. J. Math., *4* (1952), 436–444.
8. D. V. Widder, *The Laplace transform* (Princeton, 1941).

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ON A GENERALIZATION OF THE FIRST CURVATURE OF A CURVE IN A HYPERSURFACE OF A RIEMANNIAN SPACE

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1. Introduction. The unit tangent vector at a point of a curve in a hypersurface of a Riemannian space has two derived vectors along the curve, one with respect to the Riemannian space in which the hypersurface is imbedded and one with respect to the hypersurface itself. When the former vector is decomposed along the directions normal and tangent to the hypersurface, its tangential component, which is called the first curvature vector of the curve at the point in the hypersurface, is exactly the latter vector. In this respect, the first curvature at a point of a curve in a hypersurface, that is, the magnitude of the first curvature vector, is related to the unit normal vector of the hypersurface at the point. Since the normal direction to the hypersurface used in the above decomposition can be replaced by a general direction orthogonal to the curve, it is obvious that the concept of the first curvature of a curve in a hypersurface can be generalized. This note deals with such generalization and its consequences.

The notation of Eisenhart **(2)** will be used for the most part except that Γ^t_{jk} will be employed for Christoffel Symbols of the second kind.

2. Definition. Let V_n be a Riemannian space with positive definite first fundamental form $g_{ij}\,dx^i\,dx^j$ $(i, j = 1, \ldots, n)$ imbedded in a Riemannian space V_{n+1} with positive definite first fundamental form $a_{\alpha\beta}\,dy^\alpha\,dy^\beta$ $(\alpha, \beta = 1, \ldots, n+1)$. Let $C: x^i = x^i(s)$ be a curve in V_n, where s is its arc length. If q^α and p^i represent the derived vectors of the unit tangent vector t of C with respect to V_{n+1} and V_n respectively, we have

$$q^\alpha = y^\alpha{}_{,i}\,p^i + \left(\Omega_{ij}\,\frac{dx^i}{ds}\,\frac{dx^j}{ds}\right)\xi^\alpha \tag{2.1}$$

where ξ^α are the contravariant components of the unit vector normal to V_n and where Ω_{ij} the second fundamental quadratic tensor for V_n**(2**, p. 151**)**.

Let λ be unit vectors in V_{n+1}, which are not in V_n except possibly in its asymptotic directions and whose contravariant components λ^α at a point P in V_n are analytic functions of x^i and dx^i at P. The totality of these vectors λ associated with V_n is called a λ-*congruence*, which is a congruence of unit vectors if λ^α are functions of x^i only, or a congruence of hypercones of unit vectors if λ^α are functions of both x^i and dx^i, in which case we *assume* that λ is in V_n if and

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only if λ and its corresponding dx^i at x^i are coincident with an asymptotic direction in V_n. Each point of V_n is associated with one unit vector or one hypercone of unit vectors of the λ-congruence. We always assume such association between the λ-congruence and V_n in the following discussion. Expressing λ as a linear combination of independent vectors in V_{n+1}, we may write

$$\lambda^\alpha = y^\alpha{}_{,i}\, w^i + w\, \xi^\alpha \tag{2.2}$$

where w^i are the components of a contravariant vector in V_n and where w is a scalar. The case $w = 0$ which corresponds to λ in an asymptotic direction of V_n will be discussed at an appropriate place. Unless explicitly indicated w is always assumed to be different from zero. Since $a_{\alpha\beta}\lambda^\alpha\lambda^\beta = 1$, we have

$$g_{ij}\, w^i w^j + (w)^2 = 1. \tag{2.3}$$

Let N^α be a unit vector at a point P of the curve C in V_n, which satisfies the conditions: (1) it is linearly dependent on λ and the unit tangent vector t at P of C, (2) it is orthogonal to t. Hence $a_{\alpha\beta}N^\alpha N^\beta = 1$ and $a_{\alpha\beta}N^\alpha t^\beta = 0$. With the help of (2.3) and the fact that the contravariant components of t in V_{n+1} are $y^\alpha{}_{,i}\, dx^i/ds$, we have

$$N^\alpha = \pm \frac{y^\alpha{}_{,i}\{-g_{kh}w^h(dx^k/ds)(dx^i/ds) + w^2\} + w\xi^\alpha}{\{1 - g_{ij}\,g_{kh}\,w^i\,w^h(dx^j/ds)(dx^k/ds)\}^{\frac{1}{2}}}. \tag{2.4}$$

The plus sign in (2.4) is to be taken when $w > 0$, the minus sign when $w < 0$. Thus (2.4) will reduce to $N^\alpha = \xi^\alpha$ as is expected, when λ is linearly dependent on t and ξ^α; that is, $w^i = k\,dx^i/ds$, k being any constant different from unity. Elimination of ξ^α from (2.1) and (2.4) gives

$$q^\alpha = y^\alpha{}_{,i}\left(p^i - K_n\,\rho^i + K_n\,g_{kh}\,\rho^h\,\frac{dx^k}{ds}\,\frac{dx^i}{ds}\right) + N^\alpha K_n\left(1 - g_{ij}\,g_{kh}\,w^i\,w^h\,\frac{dx^j}{ds}\,\frac{dx^k}{ds}\right)^{\frac{1}{2}} \Big/ |w| \tag{2.5}$$

where K_n is the normal curvature of C and where $\rho^i = w^i/w$. Thus, when q^α is decomposed along N^α and a direction in V_n, we have from (2.5) the tangential component, $\bar{K}_g\mu^i$, defined by

$$\bar{K}_g\,\mu^i = p^i - K_n\,\rho^i + K_n\,g_{kh}\,\rho^h\,\frac{dx^k}{ds}\,\frac{dx^i}{ds} \tag{2.6}$$

where μ^i is the unit first curvature vector of C in V_n at P. We call $\bar{K}_g\mu^i$ the first curvature vector of C in V_n at P relative to the λ-congruence and $\bar{K}_g$ the first curvature of C in V_n at P relative to the λ-congruence. For convenience, they may be called the *relative first curvature vector* and the *relative first curvature* of the curve C at P respectively.

When $w = 0$, we have $w^i = dx^i/ds$ coincident with an asymptotic direction of V_n. Equations (2.4) and consequently equations (2.6) are then undefined. In this case, equations (2.1) reduce to $q^\alpha = y^\alpha{}_{,i}\,p^i$, which yield immediately

$$\bar{K}_g\,\mu^i = p^i. \tag{2.7}$$

Since equations (2.7) are derivable from (2.6) when t is asymptotic and $w \neq 0$, we see that any result obtained from (2.6) for an asymptotic t with $w \neq 0$ holds true for an asymptotic t with $w = 0$. With this understanding we may say that equations (2.6) do give us a complete knowledge of the relative first curvature vector and the relative first curvature of the curve C in a hypersurface.

A simple calculation shows that the relative first curvature of C at P is given by

$$\bar{K}_g = K_g + K_d \tag{2.8}$$

where K_g is the first curvature of C in V_n and where K_d is defined by

$$K_d = 0 \quad \text{or} \quad K_d = -K_n \rho_j \mu^j \qquad (\rho_j = g_{ij}\rho^i) \tag{2.9}$$

according as $w = 0$ or $w \neq 0$. When C is a geodesic in V_n, we have $K_g = 0$. Consequently, $\bar{K}_g = K_d$, which is then the relative first curvature of a geodesic in V_n. Hence we have

THEOREM 2.1. *The relative first curvature at a point P of a curve in V_n differs from its first curvature at P in V_n by the relative first curvature at P of the geodesic in V_n, which passes through P in the same direction as the curve.*

If t is asymptotic, we have $\bar{K}_g = K_g$ and $K_g = K$, where K is the first curvature of C in V_{n+1}. Conversely, if $K = K_g$, then t is asymptotic and $\bar{K}_g = K_g$. Hence we have

THEOREM 2.2. *The three curvatures at a point P of a curve in V_n in V_{n+1}—the relative first curvature, the first curvature in V_n, and the first curvature in V_{n+1}—are identical if and only if the direction of the curve at P is an asymptotic direction of V_n.*

From the definition of K_d in equations (2.9) it is seen that K_d vanishes if and only if: (1) t is asymptotic and $w = 0$, or (2) t is asymptotic and $w \neq 0$, or (3) t is not asymptotic and $w^i = kt$ where k is any constant different from unity. Hence we have

THEOREM 2.3. *The relative first curvature of a geodesic in V_n is zero if and only if the geodesic is an asymptotic curve, or at each point P of the geodesic, every vector of the λ-congruence associated with P is linearly dependent on the unit vector normal to V_n at P and the unit tangent vector of the geodesic at P.*

3. Pseudogeodesics. A curve in V_n is called a pseudogeodesic in V_n relative to a λ-congruence or simply a pseudogeodesic in V_n, if the relative first curvature at each point of the curve is zero. From (2.5) a curve in V_n in V_{n+1} is a pseudogeodesic in V_n if and only if the principal normal vector of the curve in V_{n+1} coincides with N^α, which according to its definition is determined by the direction of the curve and the vector of the λ-congruence. Since the osculating geodesic surface at a point of a curve in a space is the surface formed by the geodesics through the point in the pencil of directions determined by the tangent and the principal normal to the curve (**2**, p. 62), we have the following geometric property of a pseudogeodesic.

THEOREM 3.1. *A necessary and sufficient condition that a curve in V_n in V_{n+1} be a pseudogeodesic in V_n is that the osculating geodesic surface at every point of the curve considered as a curve in V_{n+1} is tangent to the unit vector of the λ-congruence associated with the curve at the corresponding point.*

Since $p^i = d^2x^i/ds^2 + \Gamma^i_{jk}(dx^j/ds)(dx^k/ds)$, we have from (2.6) that pseudogeodesics in a hypersurface with respect to the λ-congruence are defined by

$$\frac{d^2x^i}{ds^2} + \left(\Gamma^i_{jk} - \Omega_{jk}\rho^i + \Omega_{jk}\rho_m \frac{dx^m}{ds}\frac{dx^i}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{3.1}$$

These equations are n differential equations of the second order. Their complete integral involves $2n$ arbitrary constants. These may be determined by initial values of x^i and dx^i/ds, that is, values for $s = 0$, or by the initial values of x^i denoted by x^i_0 and other values of x^i such that $|x^i - x^i_0|$ are less than some fixed quantity. Hence we have

THEOREM 3.2. *Through each point and in any given direction in a hypersurface there passes a unique pseudogeodesic.*

THEOREM 3.3. *Through two sufficiently near points in a hypersurface there passes one and only one pseudogeodesic.*

By Theorem 2.2 we have

THEOREM 3.4. *If an asymptotic curve of V_n is a geodesic in V_{n+1}, it is both a geodesic in V_n and a pseudogeodesic in V_n. If a geodesic in V_n is a geodesic in V_{n+1}, it is a pseudogeodesic in V_n. An asymptotic geodesic in V_n is always a pseudogeodesic in V_n.*

By Theorem 3.1, geodesics, union curves, and hypergeodesics on a surface in ordinary space appear as special pseudogeodesics (5) in V_2 in S_3. Substitution into (2.8) and (3.1) for λ the unit normal vectors to the surface, or the unit vectors along the specified congruence in the definition of union curves, or the unit vectors along generators of the osc-cones associated with a family of hypergeodesics gives respectively the geodesic curvature of a curve and the differential equations of geodesics, or the union curvature of a curve and the differential equations of union curves (4), or the hypergeodesic curvature of a curve and the differential equations of hypergeodesics (1). *The definition of the relative first curvature of a curve has the characteristic of being independent of the differential equations of pseudogeodesics.*

4. Relative parallelism. Let C be an arbitrary curve in V_n with unit tangent vector dx^i/ds. Let v^i be contravariant components of a family of unit vectors v along C in V_n. We say that the vectors v are parallel along C relative to the λ-congruence if and only if

$$\frac{dv^i}{ds} + \left(\Gamma^i_{jk} - \Omega_{jk}\rho^i + \Omega_{jk}\rho_m \frac{dx^m}{ds}\frac{dx^i}{ds}\right)v^j\frac{dx^k}{ds} = 0. \tag{4.1}$$

Such parallelism of v along C is called a *relative parallelism*. When the λ-con-

gruence becomes a congruence of unit normal vectors to V_n, a relative parallelism of v along C reduces to parallelism of v along C in the sense of Levi-Civita.

Comparing equations (3.1) with equations (4.1), we see that the unit tangent vector to a pseudogeodesic suffers a relative parallel displacement along the pseudogeodesic. Hence we have

THEOREM 4.1. *Pseudogeodesics in V_n are relative auto-parallel curves in V_n.*

If v are not parallel along C relative to the λ-congruence, we put

$$\frac{dv^i}{ds} + \left(\Gamma^i_{jk} - \Omega_{jk}\,\rho^i + \Omega_{jk}\,\rho_m \frac{dx^m}{ds}\frac{dx^i}{ds}\right) v^j \frac{dx^k}{ds} = \nu^i. \tag{4.2}$$

Then ν^i are obviously contravariant components of a vector. We call ν^i the *relative associate curvature vector* or the *relative angular spread vector* of v along C and its magnitude $(g_{ij}\nu^i\nu^j)^{\frac{1}{2}}$ the *relative associate curvature* or the *relative angular spread* of v along C.

Let Q be a neighboring point of P on C. Let Δs be the arc length of C from P to Q. Let a unit vector v^i in V_n undergo a local displacement along C from P to Q into a unit vector $v_1|^i$ and undergo a relative parallel displacement along C from P to Q into a unit vector $v_2|^i$. It is obvious that along C dx^i/ds, its unit tangent vector, and any function of x^i are functions of s. We assume ali functions involved in the following discussion to be analytic along C in a certain common interval of s. Let g_{ij} at Q be denoted by $\bar{g}_{ij}$. Let Φ^i denote certain terms in (4.1) as follows

$$\Phi^i = \left(-\Gamma^i_{jk} + \Omega_{jk}\,\rho^i - \Omega_{jk}\,\rho_m \frac{dx^m}{ds}\frac{dx^i}{ds}\right) v^j \frac{dx^k}{ds}.$$

Then we can express $v_1|^i$, $v_2|^i$ and $\bar{g}_{ij}$ as follows in Taylor's series:

$$\begin{aligned}
v_1|^i &= v^i + \frac{dv^i}{ds}(\Delta s) + \tfrac{1}{2}\frac{d^2v^i}{ds^2}(\Delta s)^2 + \ldots,\\
v_2|^i &= v^i + \Phi^i(\Delta s) + \tfrac{1}{2}\frac{d\Phi^i}{ds}(\Delta s)^2 + \ldots,\\
\bar{g}_{ij} &= g_{ij} + \frac{dg_{ij}}{ds}(\Delta s) + \tfrac{1}{2}\frac{d^2g_{ij}}{ds^2}(\Delta s)^2 + \ldots,
\end{aligned} \tag{4.3}$$

where all the coefficients are to be evaluated at P.

Substituting (4.3) into $\bar{g}_{ij}\,v_1|^i\,v_1|^j = 1$ and $\bar{g}_{ij}\,v_2|^i\,v_2|^j = 1$, we obtain

$$\begin{aligned}
&2g_{ij}\,v^i\frac{dv^j}{ds} + \frac{dg_{ij}}{ds}\,v^iv^j = 0,\\
&g_{ij}\frac{dv^i}{ds}\frac{dv^j}{ds} + g_{ij}\,v^i\frac{d^2v^j}{ds^2} + 2\frac{dg_{ij}}{ds}\,v^i\frac{dv^j}{ds} + \tfrac{1}{2}\frac{d^2g_{ij}}{ds^2}\,v^iv^j = 0,\\
&2g_{ij}\,v^i\,\Phi^j + \frac{dg_{ij}}{ds}\,v^iv^j = 0,\\
&g_{ij}\,\Phi^i\,\Phi^j + g_{ij}\,v^i\frac{d\Phi^j}{ds} + 2\frac{dg_{ij}}{ds}\,v^i\,\Phi^j + \tfrac{1}{2}\frac{d^2g_{ij}}{ds^2}\,v^iv^j = 0.
\end{aligned} \tag{4.4}$$

Let $\Delta\theta$ be the angle between $v_1|^i$ and $v_2|^i$. Then we have

$$\cos(\Delta\theta) = \bar{g}_{ij} v_1|^i v_2|^j. \tag{4.5}$$

By Taylor's theorem and equations (4.3), (4.4) we obtain

$$\begin{aligned} \cos(\Delta\theta) &= 1 - \tfrac{1}{2}(\Delta\theta)^2 + \dots, \\ \bar{g}_{ij} v_1|^i v_2|^j &= 1 - \tfrac{1}{2} g_{ij} v^i v^j (\Delta s)^2 + \dots. \end{aligned} \tag{4.6}$$

Substitution of (4.6) into (4.5) yields

$$\left(\frac{d\theta}{ds}\right)^2 = g_{ij} v^i v^j. \tag{4.7}$$

Hence we have

THEOREM 4.2. *The relative associate curvature of a vector along a curve C at a point P is numerically the arc-rate of change of the angle between the two vectors displaced locally and relative parallelly from the vector at P along C. Such arc-rate of change of angle is zero along C if and only if the vector suffers relative parallel displacement along C.*

When $v^i = dx^i/ds$, then equation (4.7) reduces to $d\theta/ds = \pm \bar{K}_r$. Hence we have another interesting geometric interpretation of the relative first curvature.

THEOREM 4.3. *If the unit tangent vector at a point of a curve undergoes relative parallel displacement along the curve, the arc-rate of change of the angle between the vector and the curve is numerically the relative first curvature of the curve at the point.*

The content of Theorem 4.1 can be derived as a consequence of Theorem 4.3 and reads as

THEOREM 4.4. *If the unit tangent vector at a point of a pseudogeodesic undergoes relative parallel displacement along the pseudogeodesic, the arc-rate of change of the angle between the vector and the pseudogeodesic is zero at the point.*

Let the coefficients in the second term of (4.1) be denoted by $L^i{}_{jk}$, that is

$$L^i{}_{jk} = \Gamma^i{}_{jk} - \Omega_{jk} \rho^i + \Omega_{jk} \rho_m \frac{dx^m}{ds} \frac{dx^i}{ds}.$$

It is evident that if $L^i{}_{jk}$ and $\bar{L}^h{}_{pq}$ are these coefficients in different coordinate systems x^i and $\bar{x}^i$ respectively, they satisfy the equations

$$\frac{\partial^2 x^i}{\partial \bar{x}^p \partial \bar{x}^q} + L^i{}_{jk} \frac{\partial x^j}{\partial \bar{x}^p} \frac{\partial x^k}{\partial \bar{x}^q} = \bar{L}^h{}_{pq} \frac{\partial x^i}{\partial \bar{x}^h} \tag{4.8}$$

exactly the same as the coefficients of connection except that the latter are functions of x's only (3, p. 3). Hence, if these coefficients $L^i{}_{jk}$ are called the *coefficients of relative connection*, the concepts of relative parallelism and pseudogeodesics become respectively those of parallelism and paths defined in terms of the relative connection (3, pp. 13, 57).

REFERENCES

1. D. B. Dekker, *Hypergeodesic curvature and torsion*, Bull. Amer. Math. Soc., *55* (1949), 1151–1168.
2. L. P. Eisenhart, *Riemannian geometry* (Princeton, 1949).
3. ———, *Non-Riemannian geometry* (Amer. Math. Soc., 1949).
4. C. E. Springer, *Union curves and union curvature*, Bull. Amer. Math. Soc., *51* (1945), 686–691.
5. E. J. Wilczynski, *Some generalizations of geodesics*, Trans. Amer. Math. Soc., *23* (1922), 223–239.

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BROUWERIAN GEOMETRY

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1. Introduction. From time to time attention has been directed to the study of spaces in which the "distances" are selected from algebraic structures other than the real or complex numbers. For example, Menger **(9)** and Taussky **(11)** have considered spaces in which the distances are taken from a group. More recently, Ellis **(4)**, Blumenthal **(2)**, and Elliott **(3)**, have investigated spaces in which the distances are elements of a lattice. In particular, Ellis has observed that if with each pair of elements a and b of a Boolean algebra B we associate the symmetric difference $a*b = ab' + a'b$, and refer to this element as the distance between a and b, then this association satisfies (1) $a*b = b*a$, (2) $a*b = O$ if and only if $a = b$, (3) $(a*b) + (b*c) > a*c$. Thus "metrized," the Boolean algebra formally satisfies the usual postulates for a metric space. Since the points of the space as well as the distances are elements of the same lattice, Ellis has called B an autometrized Boolean algebra, while Blumenthal refers to it as a Boolean geometry. We shall use the latter term and reserve the former for more general purposes.

If with each pair of elements of an abstract set S we associate an element of a lattice L with an O such that the association satisfies conditions analogous to conditions (1), (2), and (3) for the association of $a*b$ with a and b, we refer to the structure as an L-metrized space. If S and L coincide, L will be said to be autometrized and in this case the association will be termed a metric operation. Elliott has shown that the symmetric difference is the only operation possible in a Boolean algebra which is simultaneously a metric and a group operation.

It is the aim of this paper to study certain aspects of L-metrized spaces and to extend some of the Boolean geometry to what we call Brouwerian geometries. These geometries are introduced and discussed in §2, and lead to numerous characterizations of Boolean algebras. Subgeometries of Brouwerian geometries are also considered. In §3 some general theorems on L-metrized spaces are established, and the implications of the coincidence of lattice and metric betweenness in autometrized spaces are examined. The concluding §4 contains a few results concerning the concept of congruence order together with some indications of possible future investigations.

2. Brouwerian geometry. If with each two elements a and b of a lattice L having a least element O and greatest element I there is associated a smallest

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element x such that $b + x > a$, then x is denoted by $a - b$ and the lattice is called a Brouwerian algebra (or Brouwerian logic). These algebras can readily be shown to be distributive lattices, and have been extensively studied by McKinsey and Tarski **(7)**. The element $a*b = (a - b) + (b - a)$ is called the *symmetric difference* of a and b and in case L is a Boolean algebra clearly coincides with the usual symmetric difference. Brouwerian algebras are natural extensions of Boolean algebras but are considerably wider since they include, for example, all finite distributive lattices, all chains having an O and an I, and distributive lattices with an I in which all descending chains are finite. A Brouwerian algebra is also the dual of a relatively pseudo-complemented lattice **(1)**. Other examples may be found in **(7)**.

We list below a number of elementary properties of the subtraction operation useful throughout this paper. Additional relations may be found in **(7)**.

THEOREM 2.1. *In a Brouwerian algebra the following relations hold:*

(a) $a - b < a$ (b) $a < b$ *if and only if* $a - b = O$
(c) $a - O = a$ (d) $a - a = O$ (e) $a + (b - a) = a + b$
(f) $a < b$ *implies* $a - c < b - c$ (g) $(a - b) - b = a - b$
(h) $(a + b) - c = (a - c) + (b - c)$ (i) $a - bc = (a - b) + (a - c)$
(j) $a < b$ *implies* $c - b < c - a$ (k) $(a - b) + ab = a$

The element $I - x$ is called the *Brouwerian complement* of x and is denoted by $\neg x$. Similarly $\neg\neg x = I - \neg x$. We list some of the properties of the Brouwerian complement below. Others may be found in **(7)**.

THEOREM 2.2. *In a Brouwerian algebra, the following relations hold:*

(a) $a < b$ *implies* $\neg b < \neg a$ (b) $a + \neg a = I$
(c) $\neg O = I, \neg I = O$ (d) $\neg\neg a < a$
(e) $\neg\neg\neg a = \neg a$ (f) $\neg(ab) = \neg a + \neg b$
(g) $\neg(a + b) = \neg\neg(\neg a \cdot \neg b)$ (h) $\neg(a \cdot \neg a) = I$

THEOREM 2.3. *The symmetric difference in a Brouwerian algebra is a metric operation.*

Proof. Relations (1) and (2) of the introduction are immediate, so it remains to establish (3), the "triangle inequality." By properties (i), (h), (k), and (e) of Theorem 2.1 we obtain

$$\begin{aligned} abc + (b*c) + (a*c) &= (ab)c + (c - ab) + [(a + b) - c] \\ &= c + [(a + b) - c] = a + b + c. \end{aligned}$$

Then $(a + b + c) - abc < (b*c) + (a*c)$, and by expansion of the left member using (h), (i) and (d), $(a*b) + (a*c) + (b*c) < (b*c) + (a*c)$, and thus $a*b < (b*c) + (a*c)$.

DEFINITION 2.1. A Brouwerian algebra autometrized by the symmetric difference is called a *Brouwerian geometry*.

It is often convenient to employ geometrical language and regard a triple of elements a, b, c as the vertices of a triangle with sides $a*b$, $a*c$, and $b*c$. By Theorem 2.3 the sides of any triangle in a Brouwerian geometry satisfy the triangle inequality. The notation $\Delta(a, b, c)$ will be used to designate a triangle with vertices a, b, c.

One reason why a Boolean geometry has so many novel properties is that the symmetric difference in that instance is a group operation. This is not true for Brouwerian geometries, and it is instructive to see how nearly it comes to being one in this case. Thus the group property eliminates the possibility of having isosceles triangles in a Boolean geometry. Brouwerian geometries, on the other hand, may abound with isosceles triangles, as happens, for example, in a Brouwerian chain, where every triangle is isosceles. However:

THEOREM 2.4. *A Brouwerian geometry is a Boolean geometry if and only if it is free of isosceles triangles.*

Proof. The necessity is obvious, as remarked above. If a Brouwerian geometry L is free of isosceles triangles, then $I*O = I*(x \cdot \neg x) = \neg(x \cdot \neg x) = I$ implies $x \cdot \neg x = O$. Since $x + \neg x = I$, every element has a unique complement and L is a Boolean algebra. Then $a - b = ab'$, $b - a = a'b$ and $a*b = ab' + a'b$, so L is a Boolean geometry.

COROLLARY 2.4.1. *A Brouwerian algebra is a Boolean algebra if and only if symmetric difference is a group operation.*

Proof. It is well known that symmetric difference is a group operation in a Boolean algebra. If it is a group operation in a Brouwerian algebra L, then the associated geometry contains no isosceles triangle and L is a Boolean algebra.

DEFINITION 2.2. Three elements a, b, c of any lattice are said to satisfy the *triangle inequality* if each is under the sum of the other two, and we write $(a, b, c)T$.

THEOREM 2.5. *In a Brouwerian geometry the relation $(a, b, c)T$ is equivalent to each of the relations*

(1) $a + b = a + c = b + c = a + b + c$

(2) $a - b < c,\ b - a < c,\ c - a < b$

(3) $a*b < c < a + b$

(4) $b - a = c - a,\ a - b = c - b,\ a - c = b - c$

Proof. If $(a, b, c)T$ then $a < b + c$ implies $a + b + c = b + c$, and (1) follows. Relation (2) is a result of the definition of the subtraction operation, and (3) follows from (2). To prove (4) we have $(a + b) - c = (a + c) - c = a - c = (b + c) - c = b - c$ by (1) above and by (h) of Theorem 2.1. The converse is readily established.

COROLLARY 2.5.1. *If a and b are two elements of a Brouwerian geometry then $(a, b, a*b)T$ and $ab + (a*b) = a + b$.*

Proof. These relations follow from an examination of $\Delta(O, a, b)$ and by using (1) of Theorem 2.5 and the distributive law.

THEOREM 2.6. *A Brouwerian geometry contains no equilateral triangle.*

Proof. We first establish a few properties of isosceles triangles. If $\Delta(a, b, c)$ is isosceles with $b*c = x, a*b = a*c = y$, then $(a, b, y)T$, $(a, c, y)T$ and $(b, c, x)T$. Now $(x, y, y)T$ implies $x < y$, and by (f) of Theorem 2.1, $x - c < y - c$, $x - b < y - b$, so

$$x = (b-c)+(c-b) = (x-c)+(x-b) < (y-c)+(y-b) = (a-c)+(a-b) < a.$$

Thus the base of an isosceles triangle is "under" the vertex. If $\Delta(a, b, c)$ is equilateral, then $x = y$ and $x < a$, $x < b$ together with $x + ab = a + b$ imply $ab = a + b$ or $a = b$.

DEFINITION 2.3. An ordered set of n pairwise distinct elements $(a_1, a_2, \ldots, a_n)$ of a Brouwerian geometry is called an *equilateral n-circuit* if $a_1*a_2 = a_2*a_3 = \ldots = a_{n-1}*a_n = a_n*a_1$, and $n \geqslant 3$.

The preceding theorem shows that there is no equilateral 3-circuit in a Brouwerian geometry. More generally we have:

THEOREM 2.7. *A Brouwerian geometry contains no equilateral n-circuit for n odd.*

Proof. We first show that if a, b, c, d are four elements such that $a*b = b*c = c*d = y$, and $d*a = x$, then $x = y$, so (a, b, c, d) is an equilateral 4-circuit. Let $a*c = u, b*d = v$, then $x < u + y$ and by the properties of isosceles triangles $u < y$, $u < b$ and $v < y$, $v < c$, so $x < y$. Since $(u, x, y)T$ and $(v, x, y)T$ we obtain $y = u + y = u + x$, $y = v + y = v + x$ or $y = x + uv$. From $uv < bc$, $y < x + bc$ and $y - bc < x$, or $(y - b) + (y - c) < x$. Since $(b, c, y)T$, $y - b = c - b$, $y - c = b - c$, and $(c - b) + (b - c) < x$ or $y < x$. Thus $x = y$. The theorem is now evident, since the existence of an equilateral n-circuit with n odd would imply the existence of an equilateral $(n - 2)$-circuit and hence that of an equilateral 3-circuit. Examples show that equilateral n-circuits exist for all even n.

THEOREM 2.8. *A Brouwerian geometry is a chain if and only if all triangles are isosceles.*

Proof. If every triangle of a Brouwerian geometry L is isosceles, consider $\Delta(O, a, b)$ with $a \neq b$. Then $a*b = a$ or b. Suppose $a*b = a$. Then $b < a$ and L is a chain. Conversely, if L is a chain, suppose a, b, c, are pairwise distinct with $a < b < c$. Then $b*c = c - b = c$, $a*c = c - a = c$, and every triangle is isosceles.

The preceding theorems have been closely allied to the "uniqueness of solution" property. The following show how much of the associativity of the sym-

metric difference in a Boolean geometry remains in the symmetric difference operation in a Brouwerian geometry.

DEFINITION 2.4. If x, y, z are the sides of a given triangle, then the triangle with vertices x, y, z is called the *first distance triangle* of the original. Similarly the triangle whose vertices are the sides of the first distance triangle is the *second distance triangle.*

In a Boolean geometry it follows at once from the associativity of the symmetric difference that every first distance triangle has the property of triangular fixity **(4)**. This means that each side is equal to the opposite vertex. This is no longer true for Brouwerian geometries; however, each side of a first distance triangle is under the opposite vertex, since if $x = b*c$, $y = a*c$, $z = a*b$, then $(x, y, z)T$, and $x*y < z$ by (3) of Theorem 2.5. Furthermore:

THEOREM 2.9. *In a Brouwerian geometry every second distance triangle has fixity.*

Proof. Define x, y, z as in the preceding paragraph and let $u = y*z$, $v = x*z$, $w = x*y$, $p = v*w$, $q = u*w$, $r = u*v$. It is sufficient to prove that $p = u$. We have noted that $p < u$, and next show that $u < p$. Since $(x, y, z)T$, $(x, w, y)T$ and $(x, v, z)T$, then by (4) of Theorem 2.5 we have $y - z = x - z = v - z$ and $z - y = x - y = w - y$. By (j) of Theorem 2.1 the inequality $w < z$ implies $v - z < v - w$, and $v < y$ implies $w - y < w - v$. Then $u = (y - z) + (z - y) < (v - w) + (w - v)$, or $u < p$.

THEOREM 2.10. *A Brouwerian geometry is a Boolean geometry if and only if every first distance triangle has fixity.*

Proof. The necessity has been cited earlier. Conversely, a Brouwerian geometry with the fixity property mentioned which is not a Boolean geometry would contain an isosceles triangle, say with sides x, x, y. Then $y = x*x = O$, and the triangle is degenerate.

DEFINITION 2.5. An autometrized space is called *regular* if $a*O = a$ for every element a of the space. The metric operation involved is also called regular. Clearly Brouwerian geometries and Boolean geometries are regular. It may be observed that every lattice admits the regular metric operation $a*b = a + b$, $a \neq b; a*a = O$.

DEFINITION 2.6. An autometrized space is called *distributive* if its distance lattice is distributive.

THEOREM 2.11. *In a regular distributive autometrized space* $(a*b) + ab = a + b$.

Proof. Identical with the proof of Corollary 2.5.1.

THEOREM 2.12. *A Brouwerian algebra is a Boolean algebra if and only if it admits a metric group operation. Furthermore, the operation must be the symmetric difference.*

Proof. If a Brouwerian algebra is a Boolean algebra, it admits a metric group operation, namely the symmetric difference. As Elliott **(3)** has shown, this is indeed the only metric group operation admitted. Conversely, suppose ϕ is a metric group operation in a Brouwerian algebra L. Since $O\phi O = O$, the zero element of L is the group identity, and $a\phi O = a$. From $\Delta(O, a, b)$, $a - b < a\,\phi\, b$, $b - a < a\,\phi\, b$ by (2) of Theorem 2.5, and so $a \dot{-} b < a\,\phi\, b$. Now $I\,\phi\, O = I$ and $I\,\phi(a \cdot \neg a) > I \dot{-} (a \cdot \neg a) = I$, so $a \cdot \neg a = O$. Since $a + \neg a = I$, L is a Boolean algebra, and ϕ is the symmetric difference.

This theorem supersedes Theorem 2.4 and its corollary. Since any finite distributive lattice is a Brouwerian algebra, the above result implies that any finite distributive lattice which admits a metric group operation is a Boolean algebra. Whether this is true for arbitrary distributive lattices is an open question.

DEFINITION 2.7. A *subgeometry* of an autometrized lattice is a subset with the property that the distance between each pair of its points is an element of the subset. For example, every principal ideal in a Boolean geometry is a subgeometry, as is any two elements together with their distance and the O. Thus the subgeometry of a Boolean geometry generated by two elements contains at most four points. Since subgeometries of Boolean geometries are subgroups whose elements are all of order two, it is an easy matter to describe the subgeometries generated by a finite number of elements. For the case of a Brouwerian geometry the problem is more complicated and thus far we have solved it only for the case of two elements.

THEOREM 2.13. *A subgeometry generated by two elements of a Brouwerian geometry contains at most nine elements.*

Proof. Let a and b be the given elements, and define $c = a \dot{-} b$, $d = a \dot{-} c$, $e = b \dot{-} c$, $f = b \dot{-} d$, $g = a \dot{-} e$, and $h = f \dot{-} g$. These eight elements and O will form the desired subgeometry. The calculations of the distances which arise are considerably shortened if we observe that when a and b are interchanged then c is unaltered, d and e are interchanged, f and g are interchanged, while h is unaltered.

We first prove that the points (a, b, e, d) form an equilateral 4-circuit of side c. By reason of the observation of the preceding paragraph and the discussion of equilateral 4-circuits made in Theorem 2.7, it is sufficient to show that $a \dot{-} d = c$. From $\Delta(a, b, c)$ we obtain the inequalities $d < b$, $e < a$. Since $(a, b, c)T$ and $(a, d, c)T$, then $c = (a - b) + (b - a) = (c - b) + (c - a) = c - ab$, $a \dot{-} d = (a - d) + (d - a) = (c - d) + (c - a) = c - ad$. Now $ad < ab$ implies $c - ab < c - ad < c$, or $c = a \dot{-} d$. To complete the calculation of the mutual distances for the five points, a, b, c, d, e observe that $\Delta(c, d, e)$ is the second distance triangle of $\Delta(O, a, b)$ so $c \dot{-} e = d$, $c \dot{-} d = e$, by Theorem 2.9.

The distances from f to each of the elements a, b, c, d, e will now be computed, and the distances from g found analogously. It will be shown that $f \dot{-} d = b$,

$f*c = c, f*e = e, f*b = d$ and $f*a = a$. First note that $f = b - d < b$, $g = a - e < a$. Since $(b, f, d)T$, $f*d = (f - d) + (d - f) = (b - d) + (d - f) = f + (d - f) = f + d = b + d = b$. $\Delta(a, b, d)$ being isosceles, $f < a$, and $f < c$. From $\Delta(d, f, c)$ we obtain $b*e < c - f$ or $c < c - f$. Thus $f*c = c$. Since $\Delta(b, d, e)$ is isosceles, $f < e$, and from $\Delta(c, e, f)$ we find $e = c*d < e - f$, or $f*e = e - f = e$. From $\Delta(f, b, e)$ we obtain $d = e*c < b - f$. But $b = f + d$ implies $b - f < d$, so $b*f = d$. Finally from $\Delta(f, a, e)$ we find $a = g*e < a - f$, or $f*a = a$.

The proof of the theorem is completed by computing the distances from h to each of the elements f, a, d, c, obtaining the distances to g, b, e by symmetry. It will be shown that $h*f = g$, $h*a = e$, $h*d = b$, and $h*c = c$. From $\Delta(f, e, g)$ there results $a = h + e$, $g = a - e < h$, $h < a$, and from $\Delta(f, d, g)$ the relations $b = h + d$, $f = b - d < h$, $h = f + g$, $h < b$. Now $f < e$ implies $g - e < g - f$, and $g = a - e = g - e$, so $g = g - f$. Then since $(f, g, h)T$, $g - f = h - f = g = h*f$. From $\Delta(a, f, h)$ we find $e = a*g < a - h$, and since $g < h$, $a - h < a - g$. From $(a, g, e)T$, $a - g = e - g$ so $e < a - h < e - g$, $h*a = a - h = e$. From $\Delta(f, d, h)$, $b = b*g < h*d$. Since $h*d = (h - d) + (d - h) < b$ then $h*d = b$. Finally from $\Delta(h, c, f)$ we find $c = c*g < c - h$, or $h*c = c$.

THEOREM 2.14. *The subgeometry generated by two comparable elements of a Brouwerian geometry contains at most six elements.*

Proof. Using the notation and result of Theorem 2.13, we suppose $a < b$, and prove that $f = O$, $h = g$, $d = b$. From $(a, b, c)T$, $(b, h, d)T$ and $(b, f, d)T$ there results $b = a + c = h + d = f + d = d + fh$, or $c - fh < d$. Since $c - f = c - h = c$, $c < d$, $h < c < d$ and $h - d = O$. But $f = b - d = h - d$ or $f = O$. Then $g = f*h = h$ and $d = b*f = b$.

COROLLARY 2.14.1. *If $a < b$, the six elements O, a, b, $a*b$, $b*(a*b)$ and $a*[b*(a*b)]$ form a subgeometry of a Brouwerian geometry.*

COROLLARY 2.14.2. *The six elements O, a, I, $\neg a$, $\neg\neg a$, and $a - \neg\neg a$ form a subgeometry of a Brouwerian geometry.*

Examples show that the maxima mentioned in Theorems 2.13 and 2.14 are attained. These theorems are reminiscent of a result due to Kuratowski **(6)**, who showed that the free closure algebra with one generator contains sixteen elements. A natural question to investigate would be whether a set of n elements of a Brouwerian geometry always generates a finite subgeometry. We conjecture that the subgeometry is finite and contains at most $(n + 1)^n$ elements. It is relatively easy to construct examples of Brouwerian geometries in which n elements generate a space containing precisely $(n + 1)^n$ elements.

3. General theorems: Metric betweenness. Blumenthal **(2)** has observed that a Boolean geometry is ptolemaic. A space is termed ptolemaic if the three products of the pairs of opposite distances for each four of its elements satisfy the triangle inequality. A Brouwerian geometry is also ptolemaic, but even more generally we have

THEOREM 3.1. *Every distributive L-metrized space is ptolemaic.*

Proof. Choose the notation for the six distances determined by four points so that the products of pairs of opposite distances are ax, by, cz, where $(a, b, c)T$, $(a, y, z)T$, $(b, z, x)T$, and $(c, x, y)T$. Then $a < b + c$, $x < b + z$ imply $ax < b + cz$. Also, $a < y + z$, $x < y + c$, imply $ax < y + cz$, so $ax < by + cz$. The remaining inequalities are proved analogously.

COROLLARY 3.1.1. *Every Brouwerian L-metrized space is ptolemaic.*

COROLLARY 3.1.2. *Every Boolean L-metrized space is ptolemaic.*

THEOREM 3.2. *In any L-metrized space, the sum of a pair of sides of a triangle equals the sum of any other pair of sides of the same triangle.*

Proof. The same as for (1) of Theorem 2.5.

THEOREM 3.3. *In an autometrized chain, every triangle is isosceles.*

Proof. This follows from Theorem 3.2 and the fact that any three elements of a chain are pairwise comparable.

DEFINITION 3.1 An autometrized lattice is called *symmetric* if the distance between every two of its elements is equal to the distance between their sum and product. By Theorem 2.1 it is easy to see that Brouwerian geometries (and hence Boolean geometries) are symmetric.

THEOREM 3.4. *An autometrized lattice which is symmetric and contains no isosceles triangles is distributive.*

Proof. A non-distributive lattice must contain one of the special five element sublattices discussed by Birkhoff **(1**, p. 134**)**. It is readily verified that the symmetric property implies in each case the existence of an isosceles triangle.

DEFINITION 3.2. In an L-metrized space, the element b is *metrically between* a and c if $d(a, b) + d(b, c) = d(a, c)$. The points (a, b, c) are said to be linear, and the relation is written $(a, b, c)M$.

It should be observed at once that this is not a betweenness relation in the usual sense, since it fails in many cases to have the so-called special inner point property, viz., $(a, b, c)M$ and $(a, c, b)M$ may both persist without having b and c coincide. However, for ease of locution, it is convenient to adopt the above terminology. It is clear that the relation does have the other basic betweenness property, namely, symmetry in the outer points, i.e. $(a, b, c)M$ if and only if $(c, b, a)M$.

THEOREM 3.5. *Three linear points of an L-metrized space fail to have the special inner point property if and only if they are the vertices of an isosceles triangle.*

Proof. If $\Delta(a, b, c)$ has $d(a, b) = d(a, c)$, then $d(b, c) < d(a, b)$ and $(a, b, c)M$, $(a, c, b)M$. Conversely, if these relations hold, then by Theorem 3.2, $d(a, b) = d(a, c)$.

DEFINITION 3.3. If a betweenness relation R has the property that $(a, b, c)R$ and $(a, x, b)R$ imply $(x, b, c)R$, the relation is said to have transitivity t_1. If the same two relations imply $(a, x, c)R$, the relation is said to have transitivity t_2. **(10)**.

THEOREM 3.6. *Metric betweenness has transitivity t_2.*

Proof. If $*$ denotes the metric operation, then $(a, b, c)M$ and $(a, x, b)M$ imply $(a*b) + (b*c) = a*c$ and $(a*x) + (x*b) = a*b$. Then $(a*x) + (x*b) + (b*c) = (a*c)$. Since $(x*b) + (b*c) > (x*c)$, then $a*c > (a*x) + (x*c)$, and by the triangle inequality, $a*c = (a*x) + (x*c)$, or $(a, x, c)M$. In general, metric betweenness fails to have t_1, an obvious situation in a space containing an isosceles triangle. Autometrized lattices, however, do exist in which t_1 fails for four points, no three of which determine an isosceles triangle.

DEFINITION 3.4. An element b of a lattice is *lattice between* a and c, written $(a, b, c)L$, if and only if $ab + bc = b = (a + c)(b + c)$. This relation has transitivity t_1, **(10)**, but does not, in general, have t_2. It is a betweenness relation in the usual sense. We note that in a distributive lattice, $(a, b, c)L$ if and only if $ac < b < a + c$. We shall say that metric and lattice betweenness coincide in an autometrized lattice, provided $(a, b, c)M$ if and only if $(a, b, c)L$.

THEOREM 3.7. *In an autometrized lattice L, metric and lattice betweenness coincide if and only if* (1) *metric betweenness has t_1,* (2) *L is symmetric,* (3) *$a < b < c$ implies $(a, b, c)M$.*

Proof. We employ a theorem due to Pitcher and Smiley **(10**, Theorem 10.1**)**. In verifying the hypotheses of that theorem, we observe that (1) implies that L is free of isosceles triangles and so by Theorem 3.5 that metric betweenness has the special inner point property. It is also readily shown that (2) and (3) imply $(a, a + b, b)M$ and $(a, ab, b)M$. Conversely, if metric and lattice betweenness coincide, (1) and (3) are immediate. The symmetric property is shown as follows: Consider the quadruple $(a, b, ab, a + b)$. Then $(a + b, a, ab)L$ and $(a + b, b, ab)L$ imply $[(a + b)*a] + (a*ab) = [(a + b)*b] + (b*ab) = (a + b)*ab$. Similarly, $(a, a + b, b)L$ and $(a, ab, b)L$ imply $[(a + b)*a] + [(a + b)*b] = (a*ab) + (b*ab) = a*b$. Then $(a + b)*ab = a*b$, as is evident from summing the terms.

COROLLARY 3.7.1. *An autometrized lattice in which metric and lattice betweenness coincide is distributive.*

Proof. By Theorems 3.7 and 3.4.

THEOREM 3.8. (Ellis). *In a Boolean geometry, metric and lattice betweenness coincide.*

Proof. We present a shorter version. If $(a, b, c)L$, then $ac < b < a + c$ and $b' < a' + c'$. Now $(a*b) + (b*c) = ab' + a'b + bc' + b'c = (a + c)b' + b(a' + c') < (a + c)(a' + c') = a*c$, and the triangle inequality yields equality, so that

$(a, b, c)M$. Conversely, $(a, b, c)M$ implies $(a + c)b' + b(a' + c') = ac' + a'c$. Multiplying by ac and $a'c'$ respectively, we obtain $acb' = O$ and $a'c'b = O$, so that $ac < b < a + c$. Thus $(a, b, c)L$. Theorem 3.8 is also a consequence of Theorem 3.7.

Since a non-Boolean Brouwerian geometry contains an isoceles triangle, it is evident from Theorem 3.5 that metric and lattice betweenness do not, in general, coincide in Brouwerian geometries. Indeed, the following theorems show that the coincidence of metric and lattice betweenness plus regularity serve to characterize Boolean geometries among Brouwerian geometries.

THEOREM 3.9. *An autometrized lattice with an I is a Boolean geometry if and only if it is regular, and metric and lattice betweenness coincide.*

Proof. The necessity is implied by Theorem 3.8. To show the sufficiency, let $a*I = x$, $I*x = y$, and consider the points (O, I, a, x, ax). Then $x + y = I$, $a + x = I$, and $(a, I, x)L$ imply $x + y = a*x = I$. By the symmetric property $(a*x) = (a + x)*(ax) = (I*ax) = I$. Since $I*O = I$, the absence of isosceles triangles implies $ax = O$. Thus the lattice is complemented, hence, being distributive, is a Boolean algebra. Since it is, *a fortiori*, a Brouwerian algebra, consider points (O, a, b, I), for which $a*I = a'$, $b*I = b'$. Then $(a*b) + ab = a + b$ implies $a*b < a + b$. Since $a*b < a' + b'$, it follows that $a*b < ab' + a'b$. Moreover, $(a + b) - ab < a*b$, or $(a - b) + (b - a) = ab' + a'b < a*b$. Thus $a*b$ is the symmetric difference.

COROLLARY 3.9.1. *A lattice with an I is a Boolean algebra if and only if it admits a metric group operation under which metric and lattice betweenness coincide.*

THEOREM 3.10. *A lattice with an I is a Boolean algebra if and only if it admits a metrization such that the space is of constant width and metric and lattice betweenness coincide.*

Proof. The necessity is obvious. To prove the sufficiency, consider the four elements (O, I, a, b). Then $(a*O) + (a*I) = (b*O) + (b*I) = (I*O)$. Since $a*b < (a*I) + (b*I)$, then $a*b < I*O$, and $I*O$ is the maximum distance which occurs. Since the space is of constant width, corresponding to any element a there is an element b such that $a*b = I*O$. Then $(a + b)*(ab) = I*O$, and $(a + b, ab, O)L$ implies $(a + b)*O = I*O$. Since no triangle can be isosceles, $a + b = I$, and $I*ab = I*O$ implies $ab = O$, and the lattice is a Boolean algebra.

COROLLARY 3.10.1. *A finite autometrized lattice in which metric and lattice betweenness coincide is a Boolean algebra.*

Proof. By Theorem 3.7 the space contains no isosceles triangle and must have constant width I.

Ellis has observed in **(4)** and **(5)** that the group of motions of a Boolean geometry is simply transitive. This means that for any two points a, b of the geometry, there is a motion f such that $b = f(a)$. A motion is a 1-1 mapping of

the space onto itself which preserves distance. It is clear at once that the group of motions of a Brouwerian geometry is not simply transitive, and in fact,

THEOREM 3.11. *A Brouwerian geometry is a Boolean geometry if and only if its group of motions is simply transitive.*

Proof. If a Brouwerian geometry L has a simply transitive group of motions, consider the motion carrying I into O. If $x \to y$, $I*x = \neg x = O*y$, and $x \to \neg x$. Then, $O \to I$, $x \cdot \neg x \to \neg(x \cdot \neg x) = I$, or $x \cdot \neg x = O$, and L is a Boolean geometry. The converse is evident from the remark preceding Theorem 3.11.

4. Congruence order. A space S is said to have congruence order k relative to a class of spaces M containing S, provided any space of M is congruent (isometric) to a subset of S whenever each k of its points are, and k is the smallest number with this property. This concept is due to Menger (8, p. 116) who proved, for example, that the congruence order of E_n relative to the class of metric spaces is $n + 3$. In (4) it was shown that the congruence order of a Boolean geometry relative to the class of autometrized spaces is three. We have not as yet established the congruence order of Brouwerian geometries, but the following theorems bear on this general question.

THEOREM 4.1. *If the distance function of an autometrized space is a group operation, the congruence order of the space, relative to the class of L-metrized spaces is three.*

Proof. Let L denote the autometrized space whose distance function $*$ is a group operation, and suppose S is any L-metrized space with the property that every three of its points can be congruently embedded in L. Consider any fixed element a of S, x an arbitrary element, and let $d(a, x) = u$, where $u \in L$. If $\bar{a}$ is any point of L, then there exists uniquely a point $\bar{x} \in L$ such that $\bar{a}*\bar{x} = u$. The mapping $x \to \bar{x}$ of S onto a subset of L is one to one, since if $y \in S$, $y \neq x$, and $d(a, y) = u$, then the isosceles triangle with vertices a, x, y, is congruent, by hypothesis, to an isosceles triangle in L, a contradiction. In particular, if $x = a$, then $u = O$, and $\bar{x} = \bar{a}$, so that $a \to \bar{a}$. We next prove that distances are preserved. Let $y \in S$ and $d(a, y) = v$. Then $\bar{a}*\bar{y} = v$. Suppose $d(x, y) = w$, $w \in L$. Then a triangle exists in L with sides u, v, w. However, if two sides of a triangle in L are respectively equal to two sides of another triangle in L, then by the group property, the third sides are equal. Thus, $\bar{x}*\bar{y} = w$. The example of a three point space with distances all equal to the same non-zero element of L shows that the congruence order is not two, since L contains no equilateral triangle.

COROLLARY 4.1.1 (Ellis). *The congruence order of a Boolean geometry relative to the class of L-metrized spaces is three.*

THEOREM 4.2. *A Brouwerian geometry is a Boolean geometry if and only if it has congruence order three relative to the class of L-metrized spaces.*

Proof. As a result of corollary 4.1.1, we need only show that a Brouwerian geometry L with congruence order three is a Boolean geometry. Suppose $x \cdot \neg x \neq O$ and consider the L-metrized space consisting of the four distinct elements a, b, c, d, with $a*c = b*d = x \cdot \neg x$, and the remaining distances equal to I. Each three points are congruently embeddable on the points $O, I, x \cdot \neg x$ of the Brouwerian geometry, and by hypothesis, the entire space is so embeddable. This configuration, however, is impossible in a Brouwerian geometry, for if a, b, c, d map respectively into a_1, b_1, c_1, d_1, of L, then $x \cdot \neg x < a_1, x \cdot \neg x < c_1$, by properties of isosceles triangles, and $a_1c_1 + x \cdot \neg x = a_1 + c_1$ implies $a_1c_1 = a_1 + c_1$, or $a_1 = c_1$. Thus $x \cdot \neg x = O$, a contradiction, and L is a Boolean geometry.

THEOREM 4.3. *A Brouwerian chain has congruence order four relative to the class of L-metrized spaces.*

Proof. We recall that any chain with O, I is a Brouwerian algebra and $a - b = a$ if $a > b$. Let C be such a chain metrized by the symmetric difference, i.e. a Brouwerian chain. Then (1) every triangle is isosceles, (2) if $a > b > c > d$, then $a*b = a*c = a*d = a$, $b*c = b*d = b$, $c*d = c$, (3) opposite sides of a quadruple cannot be equal, since a Brouwerian geometry contains no equilateral triangle.

Let S be any L-metrized space with the property that every four points of S are congruently embeddable in C. If S contains a point O' which is not the vertex of an isosceles triangle, map O' into the O of C, and every point x' of S into its distance x from O'. This establishes a one to one mapping $x' \to x$ of S onto a subset of C which is also a congruence, as we now prove. If x' and y' are two elements of S whose distances from O' are x and y, then $d(x', y')$ equals x or y, according as $x > y$ or $y > x$, since by hypothesis, the three points O', x', y', are congruently embeddable in C.

If S contains no point O' as described above, then every point is the vertex of an isosceles triangle. However, two isosceles triangles with the same vertex must have their legs equal, otherwise a quadruple including the vertex is determined, which maps into a quadruple in C not satisfying (3) above. We therefore map each point x' of S into x, the leg of an isosceles triangle with vertex x'. This is a one to one mapping of S onto a subset of C, since if x' and y' are distinct points of S, each the vertex of an isosceles triangle of leg x, then a quadruple including x' and y' is determined having opposite sides equal to x, a contradiction, since this implies a similar configuration in C, violating (3).

We next show that the mapping is a congruence. Consider two distinct points x', y' of S and their correspondents x, y under the mapping. We may assume $x > y$, so that $x*y = x$. Let $d(x', u') = x$, $d(y', u') = z$. Then $z \neq x$ by (3), so that $d(x', y') = x$ or z. If $d(x', y') = z$, then $z = y$ by definition of the mapping and $x < y$, a contradiction. Thus $d(x', y') = x = x*y$. We have shown that the congruence order is at most four.

To show that the congruence order is not less than four, consider a four point L-metrized space S with two opposite distances equal to a, and the remaining distances equal to b where $a < b$ and a, b are distinct elements of C. Each three points of S are congruently embeddable in C, but the entire space is not, since (3) is violated. This completes the proof of the theorem. Examples show that a Brouwerian geometry may have congruence order four without being a chain.

This paper leaves unanswered three principal problems which we are continuing to investigate, namely:

(1) How many elements are contained in a subgeometry of a Brouwerian geometry generated by more than two elements?

(2) Is an autometrized lattice with an I in which metric and lattice betweenness coincide necessarily a Boolean algebra?

(3) What is the congruence order of Brouwerian geometries relative to the class of L-metrized spaces?

References

1. Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, *25* (1948), revised edition.
2. L. M. Blumenthal, *Boolean geometry* I, National Bureau of Standards Report 1482 (1952).
3. J. G. Elliott, *Autometrization and the symmetric difference*, Can. J. Math., *5* (1953), 324–331.
4. D. O. Ellis, *Autometrized Boolean algebras* I, Can. J. Math., *3* (1951), 87–93.
5. ———, *Autometrized Boolean algebras* II, Can. J. Math., *3* (1951), 145–147.
6. C. Kuratowski, *Sur l'opération* $\bar{A}$ *de l'analysis situs*, Fund. Math., *3* (1922), 182–199.
7. J. C. C. McKinsey and A. Tarski, *On closed elements in closure algebras*, Annals of Math. *47* (1946), 122–162.
8. K. Menger, *Untersuchungen über allgemeine Metrik* II, Math. Annalen, *100* (1928), 75–163.
9. ———, *Beiträge zur Gruppentheorie* I. *Ueber einen Abstand in Gruppen*, Math. Zeitschrift, *33* (1931), 396–418.
10. Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, Trans. Amer. Math. Soc., *52* (1942), 95–114.
11. O. Taussky, *Abstracte Körper und Metrik* I, Ergebnisse eines Math. Kolloquiums, *6* (1935), 20–23.

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COMPLEXES IN ABELIAN GROUPS

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Introduction. Let G be an abelian group of order $[G] < \infty$. Let $A = \{a\}$, $B = \{b\}, \ldots$ denote non-empty finite complexes in G. Let $[A]$ be the number of elements of A. Finally put

$$A + B = \{a + b\}.$$

If $[A] + [B] > [G]$, then **(7)** obviously $A + B = G$. From now on we shall assume

0.1 $$[A] + [B] \leqslant [G].$$

A well-known theorem by Cauchy and Davenport states that

0.2 $$[A + B] \geqslant [A] + [B] - 1$$

if G is cyclic of prime order **(1; 3; 4)**. But 0.2 need not hold true any longer if G is cyclic and $[G]$ is composite. However, Chowla **(2)** proved 0.2 for cyclic G's under an additional assumption.

Let k be a fixed integer with $k \geqslant 1$. We wish to prove

0.3 $$[A + B] \geqslant [A] + [B] - k$$

and related results under various additional conditions and for arbitrary abelian G's. All these conditions Γ, $\bar{\Gamma}$, Δ, . . will be empty if $[B] \leqslant k$.

Our results can be obtained by adaptations of Davenport's method **(3)**. However, we shall use a slightly different approach which is also related to Mann's **(7)** and to another paper by the authors (appearing immediately after the present one).

The Main Result

1. The Condition Γ_1. We first prove 0.3 for complexes A, B which satisfy the following

Condition $\Gamma_1(A, B)$:

(i) *If* $[B] > k$ *and if* $\Gamma_1(A, B)$ *holds, then there is an element* b_0 *in* B *such that*

1.1 $$A + B \not\subset A + b_0.$$

(ii) $\Gamma_1(A, B)$ *implies* $\Gamma_1(A_2, B_2)$ *for every pair of complexes* A_2, B_2 *such that*

1.2 $$b_0 \subset B_2 \subset B, \quad A \subset A_2,$$

1.3 $$[A_2] - [A] = [B] - [B_2],$$

and

1.4 $$A_2 + B_2 \subset A + B.$$

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Our statement is trivial for $[B] \leqslant k$. Suppose it is proved for $[B] \leqslant n-1$ and let $[B] = n$ $(n > k)$. From 1.1 there is an a_0 in A such that

$$a_1 = a_0 + b_1 - b_0 \not\subset A$$

has solutions b_1 in B. Let $A_1 = \{a_1\}$ and $B_1 = \{b_1\}$. Since $b_0 \not\subset B_1$ we have

1.5 $$0 < [A_1] = [B_1] < [B].$$

Let $A_2 = A \cup A_1$ and let B_2 be the complement of B_1 in B. Thus 1.2 and 1.3 are satisfied and

1.6 $$0 < [B_2] < [B].$$

We now verify 1.4. Since $A + B_2 \subset A + B$, we have only to show that

1.7 $$A_1 + B_2 \subset A + B.$$

Let $a_1 \subset A_1$, $b_2 \subset B_2$. Thus $a_1 = a_0 + b_1 - b_0$. The definition of B_1 and $b_2 \subset B_2$ imply $a_0 + b_2 - b_0 \subset A$. Hence

$$\begin{aligned} a_1 + b_2 &= (a_0 + b_1 - b_0) + b_2 = (a_0 + b_2 - b_0) + b_1 \\ &\subset A + B_1 \subset A + B. \end{aligned}$$

This proves 1.7 and hence 1.4.

From 1.6 and our induction assumption, we have

1.8 $$[A_2 + B_2] \geqslant [A_2] + [B_2] - k.$$

Finally, 1.4, 1.8 and 1.3 yield

$$[A + B] \geqslant [A_2 + B_2] \geqslant [A_2] + [B_2] - k = [A] + [B] - k.$$

2. The Condition Γ_2. Our Condition Γ_1 is implied by

CONDITION $\Gamma_2([A], B)$:

(i) *If $k < [B] \leqslant [A]$, then there are two elements b_0 and b_1 in B such that*

2.1 $$[A](b_1 - b_0) \neq 0.$$

(ii) $\Gamma_2([A], B)$ *implies* $\Gamma_2([A] + [B] - [B_2], B_2)$ *for every subcomplex B_2 of B that contains b_0.*

It suffices to prove 1.1. Let $b_0 \subset B$ be arbitrary if $[B] > [A]$. Choose b_0 according to (i) if $k < [B] \leqslant [A]$. Suppose

2.2 $$a_1 = a + b - b_0 \subset A$$

for every a, b. If $[B] > [A]$, we keep a fixed and let b run through B. This would yield more than $[A]$ different elements of A. If $[B] \leqslant [A]$, we specialize $b = b_1$. If a runs through A, then so will a_1. Hence

$$\Sigma a = \Sigma a_1 = \Sigma(a + (b_1 - b_0) = \Sigma a + [A](b_1 - b_0)$$

or

$$[A](b_1 - b_0) = 0.$$

This contradicts 2.1. Thus 2.2 is false for some a, b. This implies 1.1.

3. Further specializations. Condition Γ_2 is certainly satisfied if each B_2 with

3.1 $$b_0 \subset B_2 \subset B$$

and

3.2 $$k < [B_2] \leqslant [A] + [B] - [B_2]$$

contains an element b such that

3.3 $$([A] + [B] - [B_2])(b - b_0) \neq 0.$$

This in turn is sure to be the case if the relation 3.3 has not less than $[B] - [B_2]+1$ solutions in B for each $[B_2]$ satisfying 3.2. We thus arrive at

CONDITION $\Gamma_3([A], B)$: *There is a b_0 in B such that the relation*

3.4 $$([A] + m)(b - b_0) \neq 0$$

has not less than $m + 1$ solutions b in B whenever

$$\max (0, \tfrac{1}{2}[B] - \tfrac{1}{2}[A]) \leqslant m \leqslant [B] - k - 1.$$

Condition Γ_3 is always satisfied if $[A]$ and $[B]$ are not too large. Suppose, e.g., that G has the type $(p_1^{\alpha_1}, \ldots, p_n^{\alpha_n})$ and that

3.5 $$[A] + [B] - k \leqslant \min (p_1, \ldots, p_n, [G] - k)$$

(some of the p_k's may be infinite). Then $[A] + m$ will be prime to the product of all finite p_k's and 3.4 will hold for each $b \neq b_0$. Thus 3.5 implies 0.3.

Let G be a finite cyclic group. Suppose there is a b_0 in B such that

3.6 $$b - b_0 \text{ is primitive for each } b \neq b_0.$$

Then, for $m < [B]$, 0.1 implies $[A] + m < [A] + [B] \leqslant [G]$. Hence each $b \neq b_0$ satisfies 3.4. Condition Γ_3 is satisfied, even if $k = 1$, and 0.2 holds. If we represent G by the cyclic group of residue classes (mod $[G]$), then 3.6 is equivalent to

3.7 $$(b - b_0, [G]) = 1 \text{ for each } b \neq b_0.$$

Chowla's theorem is identical with the observation that 0.1 and 3.7 imply 0.2. If $[G]$ is a prime number, then 3.7 is trivially satisfied and 0.2 follows from 0.1. This is the theorem of Cauchy and Davenport.

4. Comments. Suppose that for $k \geqslant 1$

4.1 $$[A + B] < [A] + [B] - k$$

and let $b_0 \subset B$. Then there exist two complexes A' and B' such that

4.2 $$A \subset A', \qquad b_0 \subset B' \subset B,$$

4.3 $$k < [B'] \leqslant [B],$$

4.4 $$[A'] - [A] = [B] - [B'],$$

4.5 $$A' + B' \subset A + B,$$

4.6 $$A' + B' - b_0 = A'.$$

In fact, we obtain such complexes A' and B' by iterating our construction of A_2 and B_2 (cf. §1) as often as possible, each time with b_0 as the basic element.

Let B^- be the set of elements $b' - b_0$ ($b' \subset B'$) and let B_0^- be the subgroup generated by B^-. Hence

4.7 $$k + 1 \leqslant [B'] = [B^-] \leqslant [B_0^-].$$

By 4.6, $A' + B^- = A'$. Because A' is finite, we have that B_0^- is finite and

4.8 $$A' + B_0^- = A'.$$

Thus A' consists of cosets of B_0^-. In particular, $[A'] = [A] + [B] - [B']$ will be a multiple of $[B_0^-]$. Therefore, putting $[B] - [B'] = m$,

$$([A] + m)(b - b_0) = 0$$

for each of the $[B'] = [B] - m$ elements in B'. Thus the relation

$$([A] + m)(b - b_0) \neq 0$$

has at most m solutions in B. By $k < [B'] \leqslant [B]$, we have

$$0 \leqslant m \leqslant [B] - k - 1.$$

Moreover, 4.6 implies $[B'] \leqslant [A'] = [A] + [B] - [B']$, that is,

$$m \geqslant -\tfrac{1}{2}[A] + \tfrac{1}{2}[B].$$

Consequently, if 4.1 holds, Condition Γ_3 cannot be true, which yields a second proof that Γ_3 implies 0.3.

Another consequence of 4.1 is:

$$A + B_0^- + b_0 \subset A' + B_0^- + b_0 = A' + b_0 \subset A' + B' \subset A + B.$$

Therefore:

The inequality 4.1 *implies, for each* b_0 *in* B, *the existence of at least* k *different elements* $b_i \neq b_0$ *in* B *such that the group* B_0^- *generated by the differences* $b_i - b_0$ *is finite and satisfies*

$$A + B_0^- + b_0 \subset A + B.$$

As an easy consequence of the special case $k = 1$, $0 \subset B \subset A$, $b_0 = 0$, we obtain a theorem due to Shepherdson (**8**, p. 85).

Variants

In the following, $A, B, \ldots$ will still be non-empty complexes in G. Their finiteness, however, and 0.1 will not necessarily be assumed. We wish to discuss some variants of §1. The analogues of §§2–4 being rather obvious, only some of them will be stated.

5. The complex $\bar{A}$. Let $\bar{A}$ be the complement of A in G. In this section, $\bar{A}$ and B are assumed to be finite. Only the case $[G] = \infty$ will be of interest.

If $g \subset \overline{A+B}$, then $g - b \subset \bar{A}$ for any b. Hence

$$[\bar{A}] \geqslant [\overline{A+B}].$$

In particular, the finiteness of $\bar{A}$ implies that of $\overline{A+B}$. Also, *if*

5.1 $$[B] > [\bar{A}],$$

then $A + B = G$.

Proof. Let g be any element of G and let b range through B. From 5.1, not all of the $[B]$ elements $g - b$ can lie in $\bar{A}$. Thus $a = g - b \subset A$ for some b, that is, $g \subset A + B$.

Again, let k be a fixed integer with $k \geqslant 1$. The following analogue of §1 can now be stated:

Suppose

5.2 $$[B] \leqslant [\bar{A}].$$

Then

5.3 $$[\overline{A+B}] \leqslant [\bar{A}] - [B] + k$$

provided that $\bar{A}$ and B satisfy some Condition Γ.

CONDITION $\Gamma_1(\bar{A}, B)$:

(i) *If* $[B] > k$ *and if* $\Gamma_1(\bar{A}, B)$ *holds true, then there is an element* b_0 *in* B *such that*

5.4 $$A + B \not\subset A + b_0.$$

(ii) $\Gamma_1(\bar{A}, B)$ *implies* $\Gamma_1(\bar{A}_2, B_2)$ *for every pair of complexes* A_2, B_2 *satisfying* 1.2, 1.4 *and*

5.5 $$[\bar{A}] - [\bar{A}_2] = [B] - [B_2].$$

The proof of the sufficiency of condition $\Gamma_1(\bar{A}, B)$ is identical with the proof in §1. Only 1.3 has to be replaced by 5.5 and 1.8 by

$$[\overline{A_2 + B_2}] \leqslant [\bar{A}_2] - [B_2] + k.$$

We note that the left hand terms of 5.5 and 1.3 both count the number of those elements of A_2 that do not lie in A.

CONDITION $\Gamma_2([\bar{A}], B)$ is obtained by replacing $[A]$ in Condition Γ_2 by $[\bar{A}]$ and $[A] + [B] - [B_2]$ by $[\bar{A}] - [B] + [B_2]$. In verifying this condition we use the fact that $\bar{A} + B \not\subset \bar{A} + b_1$ for some $b_1 \subset B$ implies 5.4 for some $b_0 \subset B$.

The following is an analogue of Γ_3.

CONDITION $\Gamma_3([\bar{A}], B)$: *There is a* b_0 *in* B *such that each of the relations*

5.6 $$([\bar{A}] - m)(b - b_0) \neq 0$$

has not less than $m + 1$ *solutions* b *in* B $(m = 0, 1, \ldots, [B] - k - 1)$.

6. Inversion and differences of complexes. In this section, A, B, C may be arbitrary complexes in G. They may be empty or infinite.

The *difference* $A - B$ of A and B is defined **(5)** to be the set of all those $c \subset G$ such that $c + B \subset A$. If $A \subset A'$ and $B \subset B'$ then

6.1 $$A - B' \subset A - B \subset A' - B.$$

Obviously

6.2 $$A + B \subset C \leftrightarrow A \subset C - B.$$

Another connection between sums and differences can be obtained by means of a concept essentially due to Khintchine **(6)**. Let i be any fixed element of G. The inversion $\tilde{A}$ of A with respect to i consists of all the elements $i - a$ where $a \subset A$. Thus $(\tilde{A})^{\sim} = A$ and $[\tilde{A}] = [A]$. We readily verify **(5)**

6.3 $$A - B = (\tilde{A} + B)^{\sim}, \quad A + B = (\tilde{A} - B)^{\sim}.$$

If $A + B \subset C$, then $\tilde{C} \subset (A + B)^{\sim} = \tilde{A} - B$ and hence from 6.2

6.4 $$B + \tilde{C} \subset \tilde{A}.$$

This is an analogue of Khintchine's inversion formula **(6)**.

7. The dual theorems. Formula 6.3 enables us to derive duals of §§1–4 from §5.

Let C and B denote finite non-empty complexes in G. Put

7.1 $$A = \tilde{C}.$$

Then

7.2 $$C - B = (A + B)^{\sim},$$

7.3 $$[\overline{A}] = [C] < \infty \text{ and } [C - B] = [\overline{A + B}] < \infty.$$

If $[B] > [C]$, $C - B$ is empty. Furthermore, $[C] = [G]$ implies $C = C - B = G$ on account of $[B] > 0$.

7.1–7.3 enable us to translate §5. Let k be a fixed integer with $k > 1$.

Suppose

7.4 $$[B] \leqslant [C] < [G].$$

Then

7.5 $$[C - B] \leqslant [C] - [B] + k$$

provided that B and C satisfy a Condition Δ.

Condition $\Gamma_1(\tilde{A}, B)$ yields

CONDITION $\Delta_1(C, B)$:

(i) *If $[B] > k$ and if $\Delta_1(C, B)$ holds there is an element b_0 in B such that*

7.6 $$C + B \not\subset C + b_0.$$

(ii) *$\Delta_1(C, B)$ implies $\Delta_1(C_2, B_2)$ for every pair of complexes C_2, B_2 such that*

7.7 $$b_0 \subset B_2 \subset B, \quad C_2 \subset C,$$

7.8 $$[C] - [C_2] = [B] - [B_2],$$

and

7.9 $$C - B \subset C_2 - B_2.$$

Condition Γ_3 leads to

CONDITION $\Delta_2([C], B)$: *There is a b_0 in B such that each of the relations*

7.10 $$([C] - m)(b - b_0) \neq 0$$

has not less than $m + 1$ solutions in B ($m = 0, 1, \ldots, [B] - k - 1$).

If B and $\bar{C}$ are finite, we may obtain similar results for $[\overline{C - B}]$ applying §1 rather than §5.

8. A condition on $A + B$. In the last sections of this paper, A and B denote again finite non-empty complexes in G which satisfy 0.1. Formula 6.4 suggests that the following variant of Γ_1 implies 0.3.

CONDITION $\Gamma_4(A, B)$:

(i) *If $[B] > k$ and if $\Gamma_4(A, B)$ holds, then there is an element b_0 in B such that*

8.1 $$(A + B) + B \not\subset (A + B) + b_0.$$

(ii) $\Gamma_4(A, B)$ *implies* $\Gamma_4(A, B_2)$ *for every complex B_2 such that*

8.2 $$b_0 \subset B_2 \subset B.$$

We wish to give a direct proof by induction. By 8.1, there is a $\bar{c}_0 \subset \overline{A + B}$ such that

8.3 $$c_1 + b_1 = \bar{c}_0 + b_0$$

has solutions $c_1 \subset A + B$, $b_1 \subset B$. Put $B_1 = \{b_1\}$, $C_1 = \{c_1\}$. Thus

8.4 $$0 < [C_1] = [B_1] < [B].$$

Let B_2 be the complement of B_1 in B and let C_2 denote **(3)** the complement of C_1 in $A + B$. From 8.4,

8.5 $$[C_2] = [A + B] - [B] + [B_2].$$

We readily verify (cf. 1.7) that

8.6 $$A + B_2 \subset C_2.$$

Since $b_0 \subset B_2 \subset B$ and $[B_2] < [B]$, our induction assumption implies

8.7 $$[A + B_2] \geqslant [A] + [B_2] - k.$$

Finally, 8.5, 8.6 and 8.7 yield 0.3.

9. Final corollaries. A condition which does not involve $A + B$ is

CONDITION $\Gamma_k([A], B)$:

(i) *If* $[B] > k$ *and if* $\Gamma_k([A], B)$ *holds, there are two elements* b_0, b_1 *in* B *such that*

$$([A] + [B] - k - 1)(b_1 - b_0) \neq 0. \tag{9.1}$$

(ii) $\Gamma_k([A], B)$ *implies* $\Gamma_k([A], B_2)$ *for every subcomplex* B_2 *of* B *that contains* b_0.

Proof. Suppose there exists a smallest positive integer n such that 0.3 is false for $[B] = n$. Then $n > k$ and there are two complexes A, B which satisfy Condition $\Gamma_k([A], B)$ and $[B] = n$ but not 0.3. Thus

$$[A + B] = [A] + [B] - k - 1. \tag{9.2}$$

On account of part (i) of Condition Γ_k, the relation

$$[A + B](b_1 - b_0) \neq 0 \tag{9.3}$$

then has solutions b_0, b_1 in B. This easily implies (cf. §2) that b_0 is a solution of 8.1. Therefore, we can construct a pair of sets B_2, C_2 for which 8.4, 8.5, and 8.6 hold. Moreover, by induction, 8.7 is true. This yields 0.3, contradicting 9.2.

The following is a special case of Γ_k.

CONDITION $\Gamma_k([A], B)$: *There is a* b_0 *in* B *such that*

$$([A] + m)(b - b_0) = 0 \tag{9.4}$$

has not more than $m + k$ *solutions* b *in* B $(m = 0, 1, \ldots, [B] - k - 1)$.

In a similar way, more conditions Γ and Δ can be derived.

REFERENCES

1. A. L. de Cauchy, *Recherches sur les nombres*, Œuvres (2nd series, Paris, 1882), vol. 1, 39–63.
2. I. Chowla, *A theorem on the addition of residue classes*, Proc. Indian Acad. Sci., *2* (1935), 242–243.
3. H. Davenport, *On the addition of residue classes*, J. London Math. Soc., *10* (1935), 30–32.
4. ———, *A historical note*, J. London Math. Soc., *22* (1947), 100–101.
5. H. Hadwiger, *Minkowskische Addition und Subtraktion beliebiger Punktmengen und die Theoreme von Erhard Schmidt*, Math. Zeitschr., *53* (1950), 210–218.
6. A. Khintchine, *Zur additiven Zahlentheorie*, Matem. Sbornik, *39* (1932), 27–34.
7. H. B. Mann, *On products of sets of group elements*, Can. J. Math., *4* (1952), 64–66.
8. J. C. Shepherdson, *On the addition of elements of a sequence*, J. London Math. Soc., *22* (1947), 85–88.

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ON SUMS OF SETS OF INTEGERS

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1. Introduction. Small italics denote integers. Let $A, B, \ldots$ be sets of non-negative integers. Let $A(h)$ be the number of positive integers in A that are not greater than h. Finally let $A + B$ denote the set of all integers of the form $a + b$ where $a \subset A$, $b \subset B$. The following result is implicitly contained in Mann's Proposition 11 **(4)**:

THEOREM 1. *Let* $n > 0$ *and*

$$0 \subset A, \quad 0 \subset B, \quad n \not\subset C = A + B. \tag{1.1}$$

Then there exists an m *such that*

$$C(n) - C(n - m) \geqslant A(m) + B(m), \tag{1.2}$$

$$0 < m \leqslant n, \tag{1.3}$$

$$m \not\subset C, \tag{1.4}$$

especially

$$m \not\subset A \quad \text{and} \quad m \not\subset B. \tag{1.5}$$

Finally, $a + n - m \subset C$ *for every* $a \subset A$, $a \leqslant m$.

In this paper, we prove several theorems related to Theorem 1. Like Theorem 1, each of them readily implies Mann's famous result: *Let* $n \geqslant 0$, $\gamma \leqslant 1$; $0 \subset A$, $0 \subset B$, $C = A + B$ *and*

$$A(k) + B(k) \geqslant \gamma k \qquad (k = 1, 2, \ldots, n).$$

Then

$$C(n) \geqslant \gamma n.$$

2. Khintchine's inversion principle. Let $n > 0$ be an arbitrary but fixed integer and let I be the set of the non-negative integers $\leqslant n$. Let $A, B, \ldots$ denote subsets of I. Put

$$A \oplus B = (A + B) \cap I. \tag{2.1}$$

Following Hadwiger, we define the difference $C \ominus A$ of C and A as the set of all the $d \subset I$ such that $A \oplus d \subset C$ **(2)**. Thus $C \ominus A$ is the largest subset D of I such that $A \oplus D \subset C$. Obviously

$$A \oplus B \subset C \leftrightarrow B \subset C \ominus A. \tag{2.2}$$

The inversion $\bar{A}$ of A is defined to be the set of all the integers $n - \bar{a} \subset I$ where $\bar{a} \not\subset A$ **(3)**. Thus

$$(\bar{A})^{-} = A. \tag{2.3}$$

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If $n \not\subset A$, then $0 \subset \bar{A}$; and if $0 \subset A$, then $n \not\subset \bar{A}$. We readily verify

$$C \ominus A = D \leftrightarrow A \oplus \bar{C} = \bar{D} \tag{2.4}$$

and hence, by (2.3),

$$\bar{A} \ominus \bar{C} = (\bar{C} \oplus A)^{-} = (A \oplus \bar{C})^{-} = C \ominus A. \tag{2.5}$$

Furthermore, from (2.2) and (2.4),

$$A \oplus B \subset C \leftrightarrow A \oplus \bar{C} \subset \bar{B}. \tag{2.6}$$

This is a slightly modified version of Khintchine's Inversion Formula **(3)**. It enables us to deduce new results from given ones.

We note that

$$\bar{C}(k) = k - C(n-1) + C(n-k-1) \quad (0 \leqslant k \leqslant n-1) \tag{2.7}$$

and

$$\bar{C}(n) = n - 1 - C(n-1) \qquad \text{if } 0 \subset C. \tag{2.8}$$

3. The dual of Mann's theorem. Using the above notations, Mann's theorem can be reformulated as follows:

THEOREM 1A. *Let*

$$A \subset I, \quad B \subset I, \quad C = A \oplus B \tag{3.1}$$

and suppose

$$0 \subset A, \quad 0 \subset B, \quad n \not\subset C. \tag{3.2}$$

Then there exists an m such that

$$C(n) - C(n-m) \geqslant A(m) + B(m), \tag{3.3}$$

$$0 < m \leqslant n, \tag{3.4}$$

$$m \not\subset C, \tag{3.5}$$

and

$$n - m \subset C \ominus A. \tag{3.6}$$

We note once more that (3.5) and (3.2) imply

$$m \not\subset A, \quad m \not\subset B \tag{3.7}$$

and that (3.6) and (3.2) yield

$$n - m \subset C. \tag{3.8}$$

Applying Khintchine's Inversion Formula to Theorem 1A, we obtain

THEOREM 1B. *Let*

$$A \subset I, \quad B \subset I, \quad A \oplus B \subset C \subset I \tag{3.9}$$

and assume (3.2). *Then there exists an m satisfying* (3.3), (3.4), (3.6) *and*

$$n - m \subset C \ominus B. \tag{3.10}$$

Again (3.6), (3.10), and (3.2) will imply (3.7) and (3.8).

Proof. Put

(3.11) $$D = C \ominus A.$$

Thus by (3.9) and (2.2)

(3.12) $$B \subset D$$

and by (2.4)

(3.13) $$\bar{C} \oplus A = \bar{D}.$$

From (3.2) and (3.12) we have

(3.14) $$0 \subset \bar{C}, \quad 0 \subset A, \quad n \not\subset \bar{D}.$$

By Theorem 1A, there exists therefore a number m satisfying (3.4) such that

(3.15) $$\bar{D}(n) - \bar{D}(n-m) \geqslant \bar{C}(m) + A(m),$$

(3.16) $$m \not\subset \bar{D},$$

and

(3.17) $$n - m \subset \bar{D} \ominus \bar{C}.$$

Here, (3.16) is equivalent to (3.6). Furthermore, (3.17), (2.5) and (3.12) imply

$$n - m \subset \bar{D} \ominus \bar{C} = C \ominus D \subset C \ominus B,$$

i.e. (3.10). Hence we also have (3.7) and (3.8). It remains to verify (3.3).

Since $0 \subset B \subset D \subset C$, (3.15) implies on account of (2.7) and (2.8)

(3.18) $$C(n-1) - C(n-m-1) \geqslant A(m) + D(m-1) + 1$$

if $0 < m < n$, and

(3.19) $$C(n-1) \geqslant A(n) + D(n-1)$$

if $m = n$. By (3.7), we have $m \not\subset B$. Hence (3.18) and (3.12) yield

$$C(n) - C(n-m) \geqslant C(n-1) - C(n-m-1) - 1 \geqslant A(m) + D(m-1)$$
$$\geqslant A(m) + B(m-1) = A(m) + B(m)$$

if $0 < m < n$. If $m = n$, then (3.19), (3.12) and $m = n \not\subset B$ imply

$$C(n) \geqslant C(n-1) \geqslant A(n) + D(n-1) \geqslant A(n) + B(n-1) = A(n) + B(n);$$

q.e.d.

4. Analogues of Mann's theorem. Theorem 1B can be improved slightly:

THEOREM 1C. *Under the assumptions of Theorem* 1B *there exists an* m *satisfying* (3.3), (3.6), (3.10) (and therefore also (3.7) and (3.8)) *and*

(4.1) $$m = n, \quad \text{or} \quad 0 < m < \tfrac{1}{2}n.$$

Applying the Inversion Principle to Theorem 1C, we obtain a corresponding extension of Theorem 1A (cf. §5, Remark (vii), below).

We shall also prove

THEOREM 2A. *Suppose A, B, C satisfy* (3.9),

$$0 \subset A, \quad 0 \subset B, \tag{4.2}$$

and

$$C(n) < A(n) + B(n). \tag{4.3}$$

Then there exists an m satisfying (3.4) *such that*

$$C(n) - C(n-m) \geqslant A(m) + B(m) - 1, \tag{4.4}$$

$$m \subset A, \quad m \subset B, \tag{4.5}$$

and that

$$\lambda m \subset C \ominus A \quad \text{and} \quad \lambda m \subset C \ominus B \tag{4.6}$$

for every integer λ such that $\lambda m \subset I$.

Define for any $D \subset I$

$$\epsilon(D) = \begin{cases} 1 \text{ if } 0 \subset D, \\ 0 \text{ if } 0 \not\subset D. \end{cases} \tag{4.7}$$

Thus

$$\bar{D}(n) = n - D(n-1) - \epsilon(D). \tag{4.8}$$

Replacing A, B, C consecutively by $B, \bar{C}, \bar{A}$, we deduce from Theorem 2A

THEOREM 2B. *Suppose A, B, C satisfy* (3.9),

$$0 \subset B, \quad n \not\subset C, \tag{4.9}$$

and

$$C(n) < A(n) + B(n) - (\epsilon(C) - \epsilon(A)). \tag{4.10}$$

(Obviously $0 \leqslant \epsilon(A) \leqslant \epsilon(C) \leqslant 1$.) *Then there exists an m satisfying* (3.4) *such that*

$$C(n) - C(n-m) \geqslant A(m-1) + B(m-1) + \epsilon(A), \tag{4.11}$$

$$m \subset B, \quad n - m \not\subset C, \tag{4.12}$$

and

$$\lambda m \subset C \ominus A, \quad n - \lambda m \not\subset A \oplus B \tag{4.13}$$

for every integer λ such that $\lambda m \subset I$.

We note that $m = 1$ implies $C = I$ in Theorem 2A. In 2B it implies that A is empty (cf. (4.6) and (4.13)).

Let $m = n$. Then $C(n) = A(n) + B(n) - 1$ and $n \subset B$ in both theorems. Furthermore $n \subset A$ in Theorem 2A but $n \not\subset A$, $0 \not\subset A$, $0 \not\subset C$ in Theorem 2B.

5. Generalizations to ordered groups. An ordered group is an (additively written) commutative group $G = \{g, g', \ldots\}$ with a transitive ordering such that $g' < g''$ always implies $g + g' < g + g''$. The following examples may be of interest:

(i) G is the set of all real numbers with the ordinary addition.

(ii) G is the set of positive real numbers, their "sum" being their ordinary product.

(iii) Let $\lambda > 0$. G is the set of real numbers greater than $-1/\lambda$ and the "sum" of g and h is defined to be $g + h + \lambda gh$.

(iv) G is the set of real vectors $(r_1, \ldots, r_m)$ with the ordinary addition and a lexicographic ordering.

Let $n \subset G$ be given; $n > 0$. Let I be the set of all the g's with $0 \leqslant g \leqslant n$. Let $A, B, \ldots$ again denote subsets of I. Then the definitions of Section 2 and the formulas (2.2) – (2.6) will carry over. Put

$$D(g) = \sum_{\substack{0 \leqslant d \leqslant g \\ d \subset D}} 1. \tag{5.1}$$

We can now state our main results:

THEOREM I. *Let A, B, C be finite subsets of I,*

$$A \oplus B \subset C, \tag{5.2}$$

and

$$0 \subset A, \quad 0 \subset B, \quad n \not\subset C. \tag{5.3}$$

Then there exists an $m \subset G$ with the following properties:

$$C(n) - C(n - m) \geqslant A(m) + B(m), \tag{5.4}$$

$$m = n \quad \text{or} \quad 0 < 2m < n, \tag{5.5}$$

$$n - m \subset C \ominus A, \quad n - m \subset C \ominus B. \tag{5.6}$$

THEOREM II. *Let A, B, C be finite subsets of I,*

$$A \oplus B \subset C, \tag{5.7}$$

$$0 \subset A, \quad 0 \subset B, \tag{5.8}$$

and

$$C(n) < A(n) + B(n). \tag{5.9}$$

Then there exists an $m \subset G$ with the following properties:

$$C(n) - C(n - m) \geqslant A(m) + B(m) - 1, \tag{5.10}$$

$$0 < m \leqslant n, \tag{5.11}$$

$$m \subset A, \quad m \subset B, \tag{5.12}$$

and

$$\lambda m \subset C \ominus A, \quad \lambda m \subset C \ominus B \tag{5.13}$$

for every integer λ such that $\lambda m \subset I$.

Remarks. (i) If G is the group of the ordinary integers, then the above theorems specialize to Theorems 1C and 2A respectively.

(ii) Theorem II remains valid if G is merely an ordered semi-group, i.e. a transitively ordered set with a commutative and associative addition such that $g' < g''$ always implies $g + g' < g + g''$. Furthermore G is supposed to have a null-element 0 such that $g > 0$ for every $g \neq 0$. However this extension to ordered semi-groups is only apparent since any ordered semi-group can be imbedded into an ordered group.

(iii) Both theorems remain valid if we replace (5.1) by

$$D(g) = \sum_{\substack{0<d\leq g \\ d\subset D}} f(d) \tag{5.14}$$

where $f(g)$ is any non-negative non-decreasing real-valued function in G. These generalizations can be proved along the same lines as the original theorems.

(iv) Let $\bar{A}$ denote the complement in I of a subset A of I. By applying the Inversion Principle to Theorem I, we obtain the following generalization of Mann's Theorem 1A:

THEOREM I′. *Let $\bar{A}, B, \bar{C}$ be finite subsets of I such that* (5.2) *and* (5.3) *hold true. Then there exists an $m \subset G$ satisfying* (5.5),

$$\bar{A}(m) \geqslant B(m) + (\bar{C}(n) - \bar{C}(n-m)), \tag{5.15}$$

and

$$m \not\subset A \oplus B, \quad n - m \subset C \ominus A. \tag{5.16}$$

We note that A and C need not be finite.

(v) In the same fashion, Theorem II yields the following generalization of Theorem 2B:

THEOREM II′. *Let $\bar{A}, B, \bar{C}$ be finite subsets of I satisfying* (5.7),

$$0 \subset B, \quad n \not\subset C, \tag{5.17}$$

and

$$\bar{A}(n) - \epsilon(A) < B(n) + \bar{C}(n) - \epsilon(C) \tag{5.18}$$

(*cf.* (4.7)). *Then there exists an $m \subset G$ which satisfies* (5.11),

$$\sum_{\substack{0<d\leq m \\ d\subset \bar{A}}} 1 \geqslant B(m) + \bar{C}(n) - \bar{C}(n-m) - 1, \tag{5.19}$$

$$m \subset B, \quad n - m \not\subset C, \tag{5.20}$$

and

$$\lambda m \subset C \ominus A, \quad n - \lambda m \not\subset A \oplus B \tag{5.21}$$

for every integer λ such that $\lambda m \subset I$.

(vi) Let I be finite. Then every subset D of I is finite and we have

$$\bar{D}(k) = I(k) - D(k) \tag{5.22}$$

for any $k \subset I$. Furthermore the group property of G implies

$$\sum_{0<g<m} 1 = \sum_{0<n-g<n} 1 = \sum_{0<g<m} 1 = \sum_{n-m<n-m+g<n} 1 = \sum_{n-m<g<n} 1,$$

or

$$\sum_{0<\rho<m} 1 = I(m) = I(n) - I(n-m). \tag{5.23}$$

On account of (5.22) and (5.23), we can then replace (5.15) by (5.4), (5.18) by

$$C(n) + \epsilon(C) < A(n) + B(n) + \epsilon(A), \tag{5.24}$$

and (5.19) by

$$C(n) - C(n-m) > \sum_{\substack{0<\rho<m \\ \rho \in A}} 1 + B(m) - 1. \tag{5.25}$$

(vii) The preceding remarks apply in particular when G is the additive group of the ordinary integers. In this case Theorem I′ specializes to a result containing Theorem 1A while Theorem II′ is specialized to Theorem 2B.

6. Proof of Theorem I. Since $B \subset C \ominus A$ it suffices to prove Theorem I under the stronger assumption

$$B = C \ominus A. \tag{6.1}$$

(Note that $0 \subset A$ implies $C \ominus A \subset C$. In particular, $C \ominus A$ is finite.)

Put

$$A_0 = A, \quad B_0 = B. \tag{6.2}$$

Let e_1 be the smallest element of A_0 such that

$$e_1 + b_1 + b_1' = c \begin{cases} \leqslant n \\ \not\subset C \end{cases} \tag{6.3}$$

has solutions $b_1, b_1' \subset B_0$ (if there are no such elements, then the index h of the following proof will be zero). Let B_1^* denote the set of all these solutions b_1, b_1' and let $A_1^* = e_1 \oplus B_1^*$. Thus $B_1^* \subset B_0$ while A_0 and A_1^* are disjoint. For $a_1 \subset A_1^*$ implies $a_1 = e_1 + b_1$ and hence

$$a_1 + b_1' = e_1 + b_1 + b_1' \begin{cases} \subset I \\ \not\subset C \end{cases}$$

for some $b_1, b_1' \subset B_0$. Thus $a_1 \not\subset A_0$.

Let B_1 be the complement of B_1^* in B_0 and let A_1 be the union of A_0 and A_1^*. By (6.3) we have

$$0 \not\subset B_1^*. \tag{6.4}$$

Thus

$$0 \subset A_1, \quad 0 \subset B_1. \tag{6.5}$$

LEMMA 1.

$$B_1 = C \ominus A_1.$$

Proof. By (6.1),

$$C \ominus A_1 \subset C \ominus A_0 = B_0 \tag{6.6}$$

and

$$B_1 \subset B_0. \tag{6.7}$$

If $b_1 \subset B_1^*$, then some b_1' will satisfy (6.3). Since $e_1 + b_1' \subset A_1$, (6.3) implies $b_1 \not\subset C \ominus A_1$. Thus (6.6) implies $C \ominus A_1 \subset B_1$.

Conversely, let $b_1 \subset B_0$ and $b_1 \not\subset C \ominus A_1$. Thus there is an $a_1 \subset A_1$ such that

$$a_1 + b_1 \begin{cases} \subset I \\ \not\subset C. \end{cases}$$

Since $A_0 \oplus b_1 \subset C$, we have $a_1 \subset A_1^*$ or $a_1 = e_1 + b_1'$ for some $b_1' \subset B_1^*$. Hence $a_1 + b_1 = e_1 + b_1 + b_1'$ is a solution of (6.3) and therefore $b_1 \not\subset B_1$. Thus (6.7) yields $B_1 \subset C \ominus A_1$.

We now repeat our construction as often as possible defining in the same fashion $e_2, B_2^*, A_2^*, B_2, A_2$ etc. B_0 was finite and each B_ν contains fewer elements than the preceding $B_{\nu-1}$. Thus this construction has to stop at some index $h > 0$. We then have

(6.8) $$A_h \oplus B_h \oplus B_h \subset C.$$

Moreover, by induction,

(6.9) $$B_\nu = C \ominus A_\nu,$$

(6.10) $$0 \not\subset B_\nu^*, \quad 0 \subset B_\nu \qquad (\nu = 1, 2, \ldots, h).$$

From (6.10), (6.8), and (6.9)

$$B_h \subset B_h \oplus B_h \subset C \ominus A_h = B_h.$$

Hence

(6.11) $$B_h \oplus B_h = B_h.$$

LEMMA 2.

$$e_1 < e_2 < \ldots < e_h.$$

Proof. It suffices to prove

(6.12) $$e_1 < e_2.$$

We have $e_2 \subset A_1$. If $e_2 \subset A_0$, then (6.12) follows from the minimum property of e_1 and the definition of B_1^*. But if $e_2 \subset A_1^*$, then $e_2 = e_1 + b_1$ where $b_1 \subset B_1^*$. By (6.4), $b_1 > 0$. This implies again (6.12).

By (6.10), the set B_h is not empty. Let $n - m$ be its largest element. We wish to show that m has the required properties (5.4) – (5.6).

From (6.11) and the definition of $n - m$, we have

(6.13) $$\text{either } 2(n-m) = n - m \quad \text{or } 2(n-m) > n.$$

By (5.2) and (5.3),

$$B_h \subset B = 0 \oplus B \subset A \oplus B \subset C.$$

Thus $n \not\subset C$ implies $n \not\subset B_h$ and therefore

(6.14) $$n - m \neq n.$$

(6.13) together with (6.14) yields (5.5). Obviously

$$n - m \subset B_h \subset B = C \ominus A.$$

Furthermore, $n - m \subset B_h$ implies

$$n - m \not\subset B_1^*. \tag{6.15}$$

Combining the minimum property of e_1 with (6.15), we obtain: There is no $b_1' \subset B_0$ such that

$$0 + (n - m) + b_1' \begin{cases} \subset I \\ \not\subset C \end{cases}.$$

Thus the second part of (5.6) is also verified. We prove (5.4) by means of several lemmas.

LEMMA 3.

$$B(m) = \sum_1^h B_\nu^*(m).$$

Proof. Since B is the union of the disjoint sets $B_1^*, \ldots, B_h^*, B_h$, we only have to prove

$$B_h(m) = 0. \tag{6.16}$$

Let $b \subset B_h$; $b > 0$. By (6.11),

$$b + (n - m) \subset B_h \text{ unless } b + (n - m) > n.$$

The first possibility being excluded by the maximum definition of $n - m$, we have $b > m$. This implies (6.16).

LEMMA 4.

$$C(n) - C(n - m) \geqslant A(m) + \sum_1^h A_\nu^*(m).$$

Proof. We have

$$A_h \oplus (n - m) \subset A_h \oplus B_h \subset C.$$

Thus

$$0 < a \leqslant m, \quad a \subset A_h$$

implies

$$n - m < a + (n - m) \leqslant n, \quad a + (n - m) \subset C.$$

Hence

$$C(n) - C(n - m) \geqslant A_h(m) = A(m) + \sum_1^h A_\nu^*(m)$$

since A_h is the union of the disjoint sets $A, A_1^*, \ldots, A_h^*$.

LEMMA 5.

$$A_\nu^*(m) = B_\nu^*(m) \qquad (\nu = 1, 2, \ldots, h).$$

Proof. We have $A_\nu^* = e_\nu \oplus B_\nu^*$. Thus it suffices to prove that

$$b \subset B_\nu^*, \quad 0 < b \leqslant m \tag{6.17}$$

implies $e_\nu + b \leqslant m$. Put

$$t = n - m + b. \tag{6.18}$$

Then we have to show

(6.19) $$e_\nu + t < n.$$

Case 1. $t \not\subset B_{\nu-1}$. By (6.9) there is an $a \subset A_{\nu-1}$ such that

$$a + t = a + (n - m) + b \begin{cases} \leq n \\ \not\subset C. \end{cases}$$

Since $n - m \subset B_h \subset B_{\nu-1}$ and $b \subset B_\nu^* \subset B_{\nu-1}$, the minimum property of e_ν implies $a \geq e_\nu$, and hence $e_\nu + t < a + t < n$.

Case 2. $t \subset B_{\nu-1}$. By (6.18) and (6.17), we have $t > n - m$. Thus the maximum definition of $n - m$ implies $t \not\subset B_h$. Hence $t \subset B_\mu^*$ for some μ with $\nu < \mu < h$. Thus there is a $b' \subset B_\mu^*$ such that

$$e_\mu + t + b' \begin{cases} \leq n \\ \not\subset C. \end{cases}$$

Hence by Lemma 2

$$n \geq e_\mu + t + b' > e_\mu + t \geq e_\nu + t.$$

Combining Lemmas 4, 5 and 3, we obtain (5.4).

7. Proof of Theorem II. Put

(7.1) $$A_0 = A, \quad B_0 = B.$$

Let e_1 be the smallest element of A_0 such that

(7.2) $$e_1 + b_1 = a \begin{cases} \subset I \\ \not\subset A \end{cases}$$

has solutions b_1 in B_0. (If no such elements exist, then we shall again define $h = 0$.) Let B_1^* be the set of all these solutions b_1 and let $A_1^* = e_1 \oplus B_1^*$. Thus $B_1^* \subset B_0$ while A_0 and A_1^* are disjoint. Let B_1 be the complement of B_1^* in B_0 and let A_1 be the union of A_0 with A_1^*. By (7.2),

(7.3) $$0 \not\subset B_1^*.$$

Thus, from (5.8),

(7.4) $$0 \subset A_1, \quad 0 \subset B_1.$$

Furthermore

(7.5) $$A_1^*(n) = B_1^*(n)$$

and hence, by (5.9),

(7.6) $$\begin{aligned} A_1(n) + B_1(n) &= [A_0(n) + A_1^*(n)] + [B_0(n) - B_1^*(n)] \\ &= A(n) + B(n) > C(n). \end{aligned}$$

LEMMA 1.

$$A_1 \oplus B_1 \subset C.$$

Proof. Since $A_0 \oplus B_1 \subset A_0 \oplus B_0 \subset C$, we only have to show

(7.7) $$A_1^* \oplus B_1 \subset C.$$

Let

$$\bar{a} = e_1 + b_1 \subset A_1^*, \quad b \subset B_1, \quad \bar{a} + b < n.$$

Then $0 \leqslant e_1 + b < \bar{a} + b < n$. Thus $b \subset B_0$, $b \not\subset B_1^*$ implies $e_1 + b \subset A$. Hence

$$\bar{a} + b = (e_1 + b_1) + b = (e_1 + b) + b_1 \subset A \oplus B \subset C.$$

Starting with A_1 and B_1, we define $e_2, B_2^*, A_2^*, B_2, A_2, \ldots$ in the same fashion. Since B_0 is finite and each B_ν contains fewer elements than the preceding one, our process has to stop at some index $h > 0$. Thus

$$A_h \oplus B_h \subset A_h. \tag{7.8}$$

Furthermore, by construction,

$$\left.\begin{aligned} 0 \subset A_\nu, \quad 0 \subset B_\nu, & \qquad (7.9)\\ C(n) < A_\nu(n) + B_\nu(n), & \qquad (7.10)\\ A_\nu \oplus B_\nu \subset C & \qquad (7.11) \end{aligned}\right\} \quad (\nu = 0, 1, \ldots, h)$$

(cf. (7.4), (7.6), and Lemma 1).

Since $A_h = A_h \oplus 0 \subset A_h \oplus B_h \subset A_h$, (7.8) and (7.11) imply

$$A_h = A_h \oplus B_h \subset C, \tag{7.12}$$

hence, by induction,

$$A_h \oplus \lambda B_h = A_h \subset C \tag{7.13}$$

for every integer $\lambda \geqslant 0$. Obviously,

$$B_h \subset B, \quad A \subset A_h. \tag{7.14}$$

LEMMA 2.

$$B_h \subset A \cap B \subset A \cup B \subset A_h.$$

Proof. Let $b \subset A$. Then $b \subset A \subset A_h$. If

$$b \subset B, \quad b \not\subset A, \tag{7.15}$$

then $\bar{a} = 0 + b$ is a solution of (7.2). Hence $h > 0$, $e_1 = 0$, $b \subset B_1^*$ (thus $b \not\subset B_1$), and

$$b = e_1 + b \subset A_1^* \subset A_1 \subset A_h. \tag{7.16}$$

This proves $B \subset A_h$. Since (7.15) implies $b \not\subset B_1$, it follows that $B_1 \subset A$. Thus

$$B_h \subset B_1 \subset A. \tag{7.17}$$

Using (7.14) we obtain Lemma 2.

LEMMA 3.

$$\lambda B_h \subset C \ominus A, \quad \lambda B_h \subset C \ominus B \qquad (\lambda = 0, 1, 2, \ldots).$$

Proof. By Lemma 2, and (7.13),

$$\left.\begin{matrix} A \oplus \lambda B_h \\ B \oplus \lambda B_h \end{matrix}\right\} \subset A_h \oplus \lambda B_h = A_h \subset C. \tag{7.18}$$

LEMMA 4.

$$e_1 < e_2 < \ldots < e_h.$$

Proof. It suffices to prove

$$e_1 < e_2. \tag{7.19}$$

We have $e_2 \subset A_1$. If $e_2 \subset A_0$, then (7.19) follows from the minimum property of e_1. But if $e_2 \subset A_1^*$, then $e_2 = e_1 + b_1 > e_1 + 0$ on account of (7.3).

From (7.12) and (7.10),

$$A_h(n) + B_h(n) > C(n) \geqslant A_h(n).$$

Hence $B_h(n) > 0$ and there exists a smallest positive element m in B_h. It obviously satisfies (5.11). Lemma 2 implies (5.12), and (5.13) follows from Lemma 3. We wish to show that m also satisfies (5.10).

For any finite subset D of G let $D(g \mid \text{mod } m)$ denote the number of elements d of D which are mutually incongruent (mod m) and satisfy $0 < d \leqslant g$.

LEMMA 5.

$$C(n) - C(n-m) \geqslant A_h(n \mid \text{mod } m).$$

Proof. Let $a \subset A_h$. By (7.13), each element $a + \lambda m$ which lies in I, belongs to A_h ($\lambda = 0, 1, 2, \ldots$). A_h being finite, there exists a largest element $a + \lambda_0 m$ of this kind. Thus

$$a + \lambda_0 m \leqslant n < (a + \lambda_0 m) + m$$

or

$$n - m < a + \lambda_0 m \leqslant n. \tag{7.20}$$

Conversely, our postulates for G imply that the solution λ_0 of (7.20) is unique for a given a. Thus each residue class (mod m) of A_h contains one and only one element a' with $n - m < a' \leqslant n$. Hence, by (7.12),

$$C(n) - C(n-m) \geqslant A_h(n) - A_h(n-m) = A_h(n \mid \text{mod } m).$$

LEMMA 6. *Let*

$$\left.\begin{matrix} a \subset A_{\nu-1}, & a \leqslant e_\nu + m \\ b \subset B_\nu^*, & 0 < b \leqslant m \end{matrix}\right\} \qquad (0 < \nu \leqslant h). \tag{7.21, 7.22}$$

Then

$$a \not\equiv e_\nu + b \pmod{m}. \tag{7.23}$$

Proof. Suppose (7.23) is false. Then there exists an integer λ such that

$$e_\nu + b = a + \lambda m. \tag{7.24}$$

By (7.22) and (7.21),

$$\lambda m = e_\nu + b - a > e_\nu - a \geqslant e_\nu - (e_\nu + m) = -m.$$

Thus $\lambda > -1$. Furthermore, $e_\nu + b \not\subset A_{\nu-1}$ and $a \subset A_{\nu-1}$ imply $\lambda \neq 0$. Hence $\lambda \geqslant 1$.

Since $a \subset A_{\nu-1}$ while

$$a + \lambda m = e_\nu + b \begin{cases} \subset I \\ \not\subset A_{\nu-1}, \end{cases}$$

there exists an integer μ such that

$$a + \mu m \subset A_{\nu-1}, \quad (a + \mu m) + m \begin{cases} \subset I, \\ \not\subset A_{\nu-1} \end{cases} \qquad 0 \leqslant \mu < \lambda.$$

Hence, from $m \subset B_h \subset B_\nu$ and the minimum definition of e_ν,

$$a + \mu m > e_\nu.$$

Thus (7.24) yields

$$e_\nu + b = a + \lambda m \geqslant (a + \mu m) + m > e_\nu + m.$$

This contradicts (7.22).

LEMMA 7.

$$A_h(e_h + m \,|\, \text{mod } m) \geqslant A_0(m \,|\, \text{mod } m) + \sum_1^h B_\nu^*(m).$$

Proof. Let $0 < \nu \leqslant h$. A_ν is the union of the disjoint sets $A_{\nu-1}$ and $A_\nu^* = e_\nu \oplus B_\nu^*$. By Lemma 6, $a \not\equiv a^* (\text{mod } m)$ if

$$a \subset A_{\nu-1}, \quad a \leqslant e_\nu + m, \quad a^* \subset A_\nu^*, \quad a^* \leqslant e_\nu + m.$$

Thus, each residue class (mod m) counted in $A_\nu(e_\nu + m \mid \text{mod } m)$ is counted either in $A_{\nu-1}$ $(e_\nu + m \mid \text{mod } m)$ or in A_ν^* $(e_\nu + m \mid \text{mod } m)$ but not in both. Conversely, any residue class counted in either of the latter expressions is also counted in the first one. Hence,

$$(7.25) \quad A_\nu(e_\nu + m \mid \text{mod } m) = A_{\nu-1}(e_\nu + m \mid \text{mod } m) + A_\nu^*(e_\nu + m \mid \text{mod } m).$$

Each element of A_ν^* being greater than e_ν, we have

$$(7.26) \qquad A_\nu^*(e_\nu + m \mid \text{mod } m) = A_\nu^*(e_\nu + m) = B_\nu^*(m).$$

Put $e_0 = 0$. Then, by Lemma 4, $e_\nu \geqslant e_{\nu-1}$. Hence (7.25) and (7.26) imply

$$(7.27) \quad A_\nu(e_\nu + m \mid \text{mod } m) \geqslant A_{\nu-1}(e_{\nu-1} + m \mid \text{mod } m) + B_\nu^*(m).$$

Adding (7.27) over ν, we obtain our statement.

LEMMA 8.

$$B(m) = \sum_1^h B_\nu^*(m) + 1.$$

Proof. B is the union of the disjoint sets $B_1^*, \ldots, B_h^*, B_h$. Furthermore, $B_h(m) = 1$, by the minimum definition of m.

Applying consecutively Lemmas 5, 7, and 8, we obtain

$$\begin{aligned} C(n) - C(n-m) &\geqslant A_h(n \mid \bmod m) \\ &\geqslant A_h(e_h + m \mid \bmod m) \\ &\geqslant A_0(m \mid \bmod m) + \sum_1^h B_r^*(m) \\ &= A(m) + B(m) - 1. \end{aligned}$$

This proves (5.10).

8. A variant of Theorem II. If D is any finite subset of the ordered group G, we define

$$D[g] = \sum_{\substack{0<d<g \\ d \subset D}} 1 \qquad \text{[cf. (5.1)]}.$$

THEOREM III. *Let A and B be finite subsets of G; $0 \subset A$, $0 \subset B$. Put*

$$C = A + B = \{a + b;\ a \subset A,\ b \subset B\}.$$

Let $n \subset G$, $n > 0$ and suppose

$$C[n] < A[n] + B[n]. \tag{8.1}$$

Then there exists an element $m \subset G$ with the following properties:

$$C[n] - C[n-m] \geqslant A[m] + B[m] + 1, \tag{8.2}$$

$$0 < m < n, \tag{8.3}$$

$$m \subset A, \quad m \subset B, \tag{8.4}$$

$$a + \lambda m \subset C \tag{8.5}$$

for every $a \subset A$ and every non-negative integer λ such that $a + \lambda m < n$.

Proof. Let I' denote the set of those $g \subset G$ with $0 \leqslant g < n$. Without loss of generality, we may assume that A and B are subsets of I' and replace C by the intersection of $A + B$ with I'. Replacing $I, A(g), B(g), \ldots$ by $I', A[g], B[g], \ldots$, we can readily prove Theorem III after the pattern of the proof of Theorem II.

In a similar way, a variant of Theorem I can be obtained.

The following application of Theorem III may be of interest.

THEOREM IV. *Let g^* be a positive element of G and let A and B be finite subsets of G; $0 \subset A$, $0 \subset B$. Furthermore let $\phi(g)$ be a real-valued function defined for all positive $g \subset G$ and such that $g \leqslant g' + g''$ implies $\phi(g) \leqslant \phi(g') + \phi(g'') + 1$. Finally, suppose*

$$A[h] + B[h] \geqslant \phi(h) \tag{8.6}$$

for each $h \subset G$ with $0 < h \leqslant g^*$. Then the set $C = A + B$ satisfies

$$C[h] \geqslant \phi(h) \tag{8.7}$$

for the same elements h.

Remark. Van der Corput and Kemperman **(1)** proved this result assuming only that $G = \{g, g', \ldots\}$ is an ordered set with a smallest element 0 and with a commutative and associative addition such that (i) $g + 0 = g$, (ii) $g + g' > g$ if $g' > 0$, (iii) $g' = g''$ if $g + g' = g + g''$.

Proof. It suffices to prove (8.7) for $h = g^*$.

Let H be the finite set consisting of g^* and the positive elements of C. Let $n \subset H$, $n \leqslant g^*$. Then it is sufficient to prove

$$C[n] \geqslant \phi(n) \tag{8.8}$$

assuming (8.7) for every $h \subset H$ with $h < n$.

If $C[n] \geqslant A[n] + B[n]$, then (8.8) follows from (8.6). Thus we may assume (8.1). By Theorem III, there is an $m \subset G$ that satisfies (8.2) – (8.5). By (8.2) and (8.6),

$$C[n] - C[n - m] \geqslant A[m] + B[m] + 1 \geqslant \phi(m) + 1. \tag{8.9}$$

Since $0 \subset A$, (8.5) implies $\lambda m \subset C$ for each integer $\lambda > 0$ such that $\lambda m < n$. C being finite, there is an element c_0 in C with

$$c_0 < n, \quad c_0 + m \geqslant n. \tag{8.10}$$

Let c_0 be the smallest element of C with this property. Thus $c + m < n$ if $c \subset C$, $c < c_0$. Hence

$$C[n - m] \geqslant C[c_0]. \tag{8.11}$$

Furthermore

$$C[c_0] \geqslant \phi(c_0) \tag{8.12}$$

on account of (8.10) and our induction assumption. Finally, (8.10) and the assumptions of our theorem imply

$$\phi(c_0) + \phi(m) + 1 \geqslant \phi(n). \tag{8.13}$$

Combining (8.9), (8.11), (8.12), and (8.13) we obtain (8.8).

References

1. J. G. van der Corput and J. H. B. Kemperman, *The second pearl of the theory of numbers* I. Nederl. Akad. Wetensch., Proc., *52* (1949), 696–704; or Indagationes Math. *11* (1949), 226–234.
2. H. Hadwiger, *Minkowskische Addition und Subtraktion beliebiger Punktmengen und die Theoreme von Erhard Schmidt.* Math. Z. *53* (1950), 210–218.
3. A. Khintchine, *Zur additiven Zahlentheorie.* Mat. Sbornik *39* (1932), 27–34.
4. H. B. Mann, *A proof of the fundamental theorem on the density of sums of sets of positive integers.* Ann. Math. (2), *43* (1942), 523–527.

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ON HOMOMORPHIC IMAGES OF SPECIAL JORDAN ALGEBRAS

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1. Introduction. A linear algebra is called a *Jordan algebra* if it satisfies the identities

$$ab = ba, \quad (a^2b)\,a = a^2(ba). \tag{1}$$

It is well known that a linear algebra S over a field of characteristic different from two is a Jordan algebra if there is an isomorphism $a \to a'$ of the vector-space underlying S into the vector-space of some associative algebra A such that

$$(ab)' = \tfrac{1}{2}(a' \cdot b' + b' \cdot a'),$$

where the dot denotes the multiplication in A. Such an algebra S is called a *special Jordan algebra*. As has been proved by Albert **(1)**, there exist Jordan algebras which are not special and this raises the problem of characterizing the special Jordan algebras within the class of all Jordan algebras. In particular one may ask: Do the special Jordan algebras satisfy any identity which is not a consequence of (1)? This raises the further question whether the class of special Jordan algebras can be defined by identities alone. We are concerned in this note with finding an answer to this second question.[1]

A class $\mathfrak{C}$ of abstract algebras can be defined by identities if and only if it is closed under the operations of taking subalgebras, direct unions and homomorphic images **(5)**. We call such a class a *variety* of algebras, following P. Hall. It follows from the definition that the Jordan algebras form a variety, $\mathfrak{J}$ say. Further, denote by $\mathfrak{S}$ the class of special Jordan algebras. Then it is easily shown that $\mathfrak{S}$ is closed under the operations of taking subalgebras and direct unions **(6)**, but, as we shall prove in §6, $\mathfrak{S}$ is not closed under the operation of taking homomorphic images and is therefore *not* a variety. The class of all homomorphic images of special Jordan algebras is again a variety, $\mathfrak{T}$ say; clearly $\mathfrak{T}$ is the "smallest" variety including all the special Jordan algebras and we have the trivial relation

$$\mathfrak{S} \subseteq \mathfrak{T} \subseteq \mathfrak{J}. \tag{2}$$

We shall prove in §5 that every algebra in $\mathfrak{T}$ on at most 2 generators is also in $\mathfrak{S}$, and give a criterion for deciding when a 3-generator algebra in $\mathfrak{T}$ is in $\mathfrak{S}$. With this criterion it is easy to construct algebras which are in $\mathfrak{T}$ but not in $\mathfrak{S}$;

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[1] The relation between Jordan algebras and special Jordan algebras has been completely described by Albert **(2)** in the special case of finite-dimensional semisimple algebras over a field of characteristic zero. We attack the problem from the other end by looking for partial results for the whole class of special Jordan algebras. Therefore the present work does not overlap Albert's.

we give an example in §6. Thus we obtain a Jordan algebra which is a homomorphic image of a special Jordan algebra, but which is not itself special. In §7 the peculiar difficulties of extending the methods of this note to algebras on more than 3 generators are briefly discussed, and it is shown that they cannot be extended without serious modification.

We note that the restriction on the field is essential, since over a field of characteristic two the Jordan product (in the form $xy + yx$) reduces to the Lie product: $xy - yx$, and it is well known that the Lie algebras derived in this way from associative algebras form a variety whatever the number of generators, so that $\mathfrak{S} = \mathfrak{T}$ in this case.[2]

2. $\mathfrak{T}$-algebras. In the whole of this note all algebras will be taken over a fixed but arbitrary field of characteristic $\neq 2$, so that we shall not refer to it explicitly unless necessary.

Let A be an associative algebra and define the *Jordan product* of two elements $x, y \in A$ by

$$\langle x, y \rangle = \tfrac{1}{2}(xy + yx).$$

The set A may be regarded as an algebra with respect to addition and Jordan multiplication and it will then be denoted by $\langle A \rangle$. If U is any subspace of A, we denote by $\langle U \rangle$ the subalgebra of $\langle A \rangle$ generated by U.

Now let A_0 be the free associative algebra on the free generators $x_\lambda (\lambda \in \Lambda)$. We denote by Φ the subspace spanned by the x's and refer to A_0 as the *free associative algebra on* Φ. It is clear that A_0 is uniquely determined by Φ, the x's being a basis of the space Φ. We denote by J_0 the algebra $\langle \Phi \rangle$, so that J_0 is a special Jordan algebra. The elements of J_0 are just the Jordan polynomials in the x's **(8)** and may be called the *Jordan elements* of A_0.

It is easily proved that every special Jordan algebra is a homomorphic image of J_0, for a suitable Λ, and hence every homomorphic image of a special Jordan algebra is a homomorphic image of J_0. Let us call an algebra a *$\mathfrak{T}$-algebra*, if it is a homomorphic image of a special Jordan algebra, i.e., if it is in $\mathfrak{T}$. Further we shall say that a $\mathfrak{T}$-algebra is *special* if it is a special Jordan algebra. Then we can state the result as

THEOREM 2.1.[3] *Every $\mathfrak{T}$-algebra is a homomorphic image of J_0, for a suitable Λ.*

The theorem can also be expressed by saying that J_0 is the free algebra of the variety $\mathfrak{T}$.

Let T be any $\mathfrak{T}$-algebra. By Theorem 2.1, $T \cong J_0/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of J_0; therefore T is defined up to isomorphism by J_0 and $\mathfrak{a}$. We require a criterion for deciding when T is special. Since $J_0 \subseteq A_0$, $\mathfrak{a}$ is contained in A_0, but of course it is not in general an ideal of A_0. We denote by $\{\mathfrak{a}\}$ the ideal of A_0 generated by the set $\mathfrak{a}$. Then we have

[2] This follows from the Birkhoff-Witt embedding theorem **(4; 10)**.

[3] Theorem 2.1 is practically a restatement, for the case of special Jordan algebras, of Theorem 1 in **(6)**.

THEOREM 2.2. *Let $\mathfrak{a}$ be an ideal of J_0. Then $J_0/\mathfrak{a}$ is a special Jordan algebra if and only if $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$.*

We note that since $\mathfrak{a} \subseteq J_0$ and $\mathfrak{a} \subseteq \{\mathfrak{a}\}$, we have in any case $\mathfrak{a} \subseteq \{\mathfrak{a}\} \cap J_0$, so that the theorem asserts in fact equality in case $J_0/\mathfrak{a}$ is special.

Proof. Suppose $\mathfrak{a}$ is special. Then, by definition, $J_0/\mathfrak{a}$ can be embedded in an associative algebra, A say. Denote the natural homomorphism of J_0 onto $J_0/\mathfrak{a}$ by $u \to \bar{u}$, then it is clear that the elements $\bar{x}_\lambda$ generate $J_0/\mathfrak{a}$. Now consider the mapping $\phi : x_\lambda \to \bar{x}_\lambda$. Since A_0 is free associative on the x_λ, we can extend ϕ to a homomorphism of A_0 into A, which we again denote by ϕ. Let $\mathfrak{b}$ be the kernel of the homomorphism ϕ, then $\mathfrak{b}$ is an ideal of A_0. Clearly ϕ induces a homomorphism of J_0 onto the Jordan algebra generated by the $\bar{x}_\lambda$, which is just $J_0/\mathfrak{a}$ by definition, and since ϕ maps x_λ into $\bar{x}_\lambda$, it follows that ϕ coincides on J_0 with the natural homomorphism onto $J_0/\mathfrak{a}$. We shall prove

$$\text{(i)} \quad \{\mathfrak{a}\} \subseteq \mathfrak{b}, \qquad \text{(ii)} \quad \mathfrak{b} \cap J_0 \subseteq \mathfrak{a}.$$

From this it then follows that $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$.

To prove (i) let $u \in \mathfrak{a}$, then $\bar{u} = 0$ by definition, and since ϕ coincides with the natural homomorphism on J_0, we have $u^\phi = 0$, which means that $u \in \mathfrak{b}$. Hence $\mathfrak{a} \subseteq \mathfrak{b}$, and since $\mathfrak{b}$ is an ideal of A_0 it follows that $\{\mathfrak{a}\} \subseteq \mathfrak{b}$.

To prove (ii) we take u in $\mathfrak{b} \cap J_0$, then, since $u \in \mathfrak{b}$, $u^\phi = 0$, and because $u \in J_0$ this means that u lies in the kernel of the natural homomorphism, i.e. $u \in \mathfrak{a}$. This proves (ii), and hence $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$.

Conversely, suppose that $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$; then we have equality by the remark which precedes the proof. We consider the associative algebra $A = A_0/\{\mathfrak{a}\}$. Denote by $\bar{x}_\lambda$ the image of x_λ under the natural homomorphism of A_0 onto $A_0/\{\mathfrak{a}\}$ and let J be the subalgebra of $\langle A \rangle$ generated by the $\bar{x}_\lambda$. Then J is the homomorphic image of J_0 under the natural homomorphism $A_0 \to A_0/\{\mathfrak{a}\}$ and therefore $J = J_0/J_0 \cap \{\mathfrak{a}\}$, which equals $J_0/\mathfrak{a}$ by hypothesis. Therefore $J_0/\mathfrak{a} = J$ is a special Jordan algebra, and this completes the proof.

The concept of an associative algebra and the special Jordan algebras embedded in it can be generalized to the case of an abstract algebra A and subsets of A which are closed under certain combinations of the operators of A **(6)**. Both the theorems of this section can be extended without difficulty to this general case.

3. The reversal operator. Let A_0 again be the free associative algebra on the space with the basis x_λ ($\lambda \in \Lambda$). We define the *reversal operator* j on A_0 as the linear mapping of A_0 into itself given by

$$x^j{}_\lambda = x_\lambda, \qquad \lambda \in \Lambda,$$
$$(uv)^j = v^j \cdot u^j, \qquad u, v \in A_0.$$

These equations together with linearity define j completely. It is clear that j is an involution; in fact j is essentially the fundamental involution on the uni-

versal associative enveloping algebra of J_0 **(7)**. If $u \in A_0$, then u^j will be called the *reverse* of u, and u is said to be *reversible*, if $u^j = u$. The set of all reversible elements of A_0 forms a subalgebra of $\langle A_0 \rangle$, as is easily verified.

We define a second linear mapping $u \to u^*$ in A_0 by the rule:

$$u^* = \tfrac{1}{2}(u + u^j). \tag{3}$$

If $u \in A_0$, then u^* is reversible, and $u^* = u$ if and only if u is reversible.

LEMMA 3.1. *If $u, v \in A_0$, then*

$$\langle u^*, v \rangle^* = \langle u^*, v^* \rangle. \tag{4}$$

The lemma is proved by a straightforward calculation:

$$\langle u^*, v \rangle^* = \tfrac{1}{2}\{\langle u^*, v \rangle + \langle u^*, v^j \rangle\} = \tfrac{1}{2}\langle u^*, v + v^j \rangle = \langle u^*, v^* \rangle.$$

Since the right-hand side of (4) is symmetric in u and v, we also have

$$\langle u, v^* \rangle^* = \langle u^*, v^* \rangle. \tag{5}$$

The formula (5) bears a remarkable resemblance to Baker's formula for commutators in A_0. If $u \to u^\dagger$ denotes the operation of forming left-normed commutators in the free generators of A_0 and $[u, v] = uv - vu$, then Baker's formula **(3; 9)** states

$$(uv^\dagger)^\dagger = [u^\dagger, v^\dagger] \qquad \textit{for all } u, v \in A_0.$$

This operation † leaves a homogeneous element of A_0 unchanged except for a scalar factor, if and only if it is a Lie element (i.e. a sum of commutators in the x's; see e.g. **(9)**). As we shall see in §4, the operation * defined by (3) leaves an element of A_0 involving less than four generators unchanged if and only if it is a Jordan element.

Consider again the free associative algebra A_0 on the x's. Any given element w of A_0 is obtained from the x's by the operations of A_0 and we refer to w, considered as a function of the x's, as an *associative polynomial* in the x's. If w belongs to J_0 it can be formed from the x's by the operations of J_0, and as such it is called a *Jordan polynomial* **(8)**. The usual definitions of degree and homogeneity in the x's taken together or in any one of them can then be extended to associative and Jordan polynomials. If $f(\xi_1, \ldots, \xi_n)$ is a (Jordan or associative) polynomial in n variables, we define the *reverse* of f to be the polynomial f^j, where

$$f^j(x_1, \ldots, x_n) = [f(x_1, \ldots, x_n)]^j.$$

Since the x's are free generators, this defines f^j as a polynomial. We say that f is *reversible*, if $f^j = f$.

As long as we restrict ourselves to polynomials with the free generators x_λ as arguments there is no need for the new terminology, but we shall want to use it in the case where the arguments are not x's, but certain other elements in A_0. Thus, if $u_1, u_2, \ldots, u_n$ are given elements of A_0 and f is a reversible poly-

nomial, then the element $f(u_1, \ldots, u_n)$ of A_0 is not necessarily reversible since the u's need not be so. To give an example, if $u = x_1 x_2$, then the element $x_1x_2x_3 + x_3x_1x_2$ is not reversible, but by writing it as $ux_3 + x_3u$ we can express it as a reversible polynomial in u and x_3.

LEMMA 3.2. *Let $\mathfrak{b}$ be an ideal of A_0 with a set of reversible elements u_ι ($\iota \in I$) as generators. If w is a reversible element of $\mathfrak{b}$ then w can be written as a reversible associative polynomial in the u's and x's which is linear homogeneous in the u's.*

Proof. Any element w of b can be written as $f(u, x)$, where f is a polynomial in the u's and x's in which each term has degree at least one in the u's. If any term of f contains more than one factor u, we express the surplus factors in terms of the x's, and in this way obtain w as a polynomial $g(u, x)$ which is linear homogeneous in the u's. The reverse polynomial $g^j(u, x)$ is again homogeneous linear in the u's and $\{g(u, x)\}^j = g^j(u^j, x) = g^j(u, x)$, since the u's are reversible. Using the fact that w is reversible we have

$$w = w^* = \tfrac{1}{2}\{g(u, x) + g^j(u, x)\},$$

and this is the required representation of w as a reversible associative polynomial which is linear homogeneous in the u's.

To illustrate the point of the lemma, let $u = x^2$, then $w = ux$ is the value of the associative polynomial $f(\xi, \eta) = \xi\eta$ for $\xi = u$, $\eta = x$, and f is not reversible, but w can also be written as $\frac{1}{2}(ux + xu)$ and this expresses w as a reversible associative polynomial which is linear homogeneous in u.

4. The connection between reversible and Jordan elements. Let R be the set of reversible elements of A_0. Then R is a subalgebra of $\langle A_0 \rangle$, and since R contains the generators x_λ, it contains J_0. Thus every Jordan element of A_0 is reversible.[4]

The question naturally arises under what circumstances J_0 equals R. We shall prove now that this is the case when the number of free generators of A_0 is less than four and only then. To obtain the best possible results it is convenient to suppose that the index-set Λ is totally ordered; so in order to avoid unnecessary detail we shall from now on take the free generators of A_0 to be well-ordered and write them as $x_1, x_2, \ldots$ without specifying the index-set unless this is relevant.

THEOREM 4.1. *Every reversible element of A_0 can be expressed as a Jordan polynomial in the generators $x_1, x_2, \ldots$ and the elements*

$$(x_{i_1}x_{i_2}x_{i_3}x_{i_4})^* \qquad i_1 < i_2 < i_3 < i_4;\ i = 1, 2, \ldots. \tag{6}$$

The expression (6) will be called a *tetrad*, more precisely the tetrad defined by

$$x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}.$$

[4]Since A_0 is the universal associative enveloping algebra of J_0, this is a special case of a result by Jacobson and Rickart (8).

We shall refer to (6) as a tetrad even if the indices are not ascending, but we insist on their being distinct.

We prove Theorem 4.1 by induction on the degree of the element, considered as an associative polynomial in the x's. If the reversible element w has degree n, the terms of highest degree will be of the form

$$\sum a_{i_1 \dots i_n} (x_{i_1} \dots x_{i_n})^*. \tag{7}$$

Let us denote the subalgebra of $\langle A_0 \rangle$ generated by the x's and by the elements (6) by S for the moment; it will be enough to prove (under the induction hypothesis) that

$$(x_{i_1} \dots x_{i_n})^* \in S.$$

For brevity we shall write this as a congruence

$$(x_{i_1} \dots x_{i_n})^* \equiv 0, \tag{8}$$

where the modulus S is always understood. If $n = 1$, (8) holds by definition of S. Now let $n > 1$. By the induction hypothesis,

$$(x_{i_2} \dots x_{i_n})^* \equiv 0.$$

Because S is a subalgebra of $\langle A_0 \rangle$, it follows that

$$\langle x_{i_1}, (x_{i_2} \dots x_{i_n})^* \rangle \equiv 0,$$

and so by applying Lemma 3.1, we get

$$\tfrac{1}{2}\{(x_{i_1} x_{i_2} \dots x_{i_n})^* + (x_{i_2} \dots x_{i_n} x_{i_1})^*\}$$
$$= \langle x_{i_1}, x_{i_2} \dots x_{i_n} \rangle^* = \langle x_{i_1}, (x_{i_2} \dots x_{i_n})^* \rangle \equiv 0.$$

Thus

$$(x_{i_1} \dots x_{i_n})^* \equiv -(x_{i_2} \dots x_{i_n} x_{i_1})^*. \tag{9}$$

The congruence (9) states that the left-hand side of (8) changes sign (mod S) under a cyclic permutation of the suffixes. If n is odd then by repeating this operation n times we get

$$(x_{i_1} \dots x_{i_n})^* \equiv -(x_{i_1} \dots x_{i_n})^*,$$

and hence (8). If n is even we apply Lemma 3.1 with

$$u = x_{i_1} x_{i_2}, \quad v = x_{i_3} \dots x_{i_n}$$

and obtain similarly

$$0 \equiv \langle (x_{i_1} x_{i_2})^*, \ x_{i_3} \dots x_{i_n} \rangle^*$$
$$\equiv \tfrac{1}{2}\{(x_{i_1} x_{i_2} x_{i_3} \dots x_{i_n})^* + (x_{i_3} x_{i_1} x_{i_2} \dots x_{i_n})^* + (x_{i_3} \dots x_{i_n} x_{i_1} x_{i_2})^*$$
$$+ (x_{i_3} \dots x_{i_n} x_{i_2} x_{i_1})^*\}.$$

By (9) we can apply two cyclic interchanges to each of the third and fourth terms without affecting their values (mod S). This will change them into the first and second term respectively, so that

$$\tfrac{1}{2}\{(x_{i_1}\dots x_{i_n})^* + (x_{i_2}x_{i_1}x_{i_3}\dots x_{i_n})^*\} = 0.$$

Multiplying by two and transposing, we obtain

$$(x_{i_1}x_{i_2}x_{i_3}\dots x_{i_n})^* \equiv -(x_{i_2}x_{i_1}x_{i_3}\dots x_{i_n})^*. \tag{10}$$

By (9) and (10) the left-hand side of (8) changes sign (mod S) if we apply the permutations $(1\,2\dots n)$ or $(1\,2)$ to the indices. These two permutations are odd (since n is even) and they generate the symmetric group on n letters, so it follows that if we apply any permutation to the indices on the left-hand side of (8), this has the effect of multiplying it by ± 1 (mod S) according to whether the permutation is even or odd. In particular if two indices are equal, the expression is $\equiv 0$ and (8) follows.

To complete the proof we distinguish three cases:

(i) $n = 2$. Then

$$(x_{i_1}x_{i_2})^* = \langle x_{i_1}, x_{i_2}\rangle \equiv 0.$$

(ii) $n = 4$. We need only consider the case where the indices are distinct. By at most changing the sign we can arrange the indices in any given tetrad in ascending order and then it is $\equiv 0$ by the definition of S.

(iii) $n \geqslant 6$. We apply Lemma 3.1 with

$$u = x_{i_1}x_{i_2}x_{i_3}x_{i_4},\quad v = x_{i_5}\dots x_{i_n}.$$

This gives

$$\tfrac{1}{4}\{(x_{i_1}\dots x_{i_4}x_{i_5}\dots x_{i_n})^* + (x_{i_4}\dots x_{i_1}x_{i_5}\dots x_{i_n})^* + (x_{i_3}\dots x_{i_4}x_{i_1}\dots x_{i_n})^* + (x_{i_2}\dots x_{i_4}x_{i_3}\dots x_{i_1})^*\} = 0.$$

Each of the second, third, and fourth terms differs from the first by an even permutation and hence we obtain

$$(x_{i_1}x_{i_2}\dots x_{i_n})^* \equiv 0,$$

i.e. (8). Hence the expression (7) is in S and by subtracting it from w we can reduce the degree to $n-1$. By induction it then follows that $w \in S$ and this completes the proof.

Since any tetrad of the form (6) must contain four distinct x's, there can be no such tetrads when the number of free generators is less than four. Hence we deduce

THEOREM 4.2. *If the number of free generators is less than four then every reversible element is in J_0.*

The following scholium, noted already in **(7)**, without proof, shows that Theorem 4.1 is best possible.

SCHOLIUM 4.3. *If the number of free generators of A_0 is $\geqslant 4$ and if $i_1 < i_2 < i_3 < i_4$, then*

$$(x_{i_1}x_{i_2}x_{i_3}x_{i_4})^* \notin J_0.$$

In the proof we can clearly confine ourselves to the case where the number of free generators of A_0 is just 4. Thus we have to prove that $(x_1x_2x_3x_4)^* \notin J_0$, and this will follow if we can prove it in any homomorphic image of A_0. Let A be the associative algebra on x_1, x_2, x_3, x_4 with the relations[5]

$$x_ix_j + x_jx_i = 0 \qquad i, j = 1, 2, 3, 4.$$

Then A has a basis consisting of the elements

$$x_{i_1} \ldots x_{i_r}, \qquad i_1 < \ldots < i_r, \quad r = 1, 2, 3, 4.$$

The Jordan algebra generated by the x's, J say, is spanned by x_1, x_2, x_3, x_4. But $(x_1x_2x_3x_4)^* = x_1x_2x_3x_4 \notin J$ and the scholium follows.

5. The embedding of 2- and 3-generator algebras. The theorems of §4 enable us to find necessary and sufficient conditions for $\mathfrak{T}$-algebras on at most 3 generators to be special, and to prove that every 2-generator $\mathfrak{T}$-algebra is special. Of course the case of a 1-generator $\mathfrak{T}$-algebra is trivial, since any Jordan product of degree n in a single generator x is just x^n. This follows by induction from the formula

$$\langle x^i, x^j \rangle = x^{i+j}.$$

THEOREM 5.1. *Let A_0 be the free associative algebra on the free generators x, y, z, and J_0 the special Jordan-algebra on x, y, z as before. If $u_1, u_2, \ldots$ are any elements of J_0 and $\mathfrak{a}$ is the ideal of J_0 generated by them, then $J_0/\mathfrak{a}$ is special if and only if*

$$(u_ixyz)^* \in \mathfrak{a} \qquad i = 1, 2, \ldots . \tag{11}$$

Proof. By Theorem 2.2, $J_0/\mathfrak{a}$ is special if and only if $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$. Since the number of free generators is 3, J_0 consists just of the reversible elements of A_0 (Theorem 4.2), and so $J_0/\mathfrak{a}$ is special if and only if every reversible element of $\{\mathfrak{a}\}$ is in $\mathfrak{a}$. It is clear that $(u_ixyz)^*$ is reversible and belongs to $\{\mathfrak{a}\}$, hence condition (11) is necessary. Conversely, suppose that (11) is satisfied and let w be any reversible element in $\{\mathfrak{a}\}$. By Lemma 3.2, w can be written as a reversible associative polynomial f in the u's and x, y, z which is linear homogeneous in the u's. We now regard $x, y, z, u_1, u_2, \ldots$ as independent. Because f is reversible, it can by Theorem 4.1 be expressed as a Jordan polynomial ϕ in $x, y, z, u_1, u_2, \ldots$ and tetrads involving these variables. Since f is linear in the u's, so is ϕ and therefore no tetrad can involve more than one u, but it must involve at least one, since the four arguments of a tetrad are distinct. By a permutation of the arguments any such tetrad can be reduced to the form $(u_ixyz)^*$ plus a Jordan polynomial in u_i, x, y, z. By hypothesis $(u_ixyz)^* \in \mathfrak{a}$, hence every term of ϕ has been reduced to a Jordan product with at least one factor in $\mathfrak{a}$, and it follows that ϕ itself is in $\mathfrak{a}$. Since $\phi = f = w$, and w is any reversible element of $\{\mathfrak{a}\}$, this shows that $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$ if (11) holds and this completes the proof.

[5] Thus A is the Grassmann algebra on the x's. I am indebted to one of the referees for the idea of this proof.

In the next section we shall construct an ideal which fails to satisfy this criterion and therefore defines a non-special $\mathfrak{T}$-algebra. First we deduce what happens in the case of two generators.

THEOREM 5.2. *Any $\mathfrak{T}$-algebra on two generators is special.*

Proof. We have to show that if A_0 is free associative on x, y and J_0 the special Jordan algebra on x, y, then every homomorphic image of J_0 is special. Let $\mathfrak{a}$ be any ideal of J_0 and $u_1, u_2, \ldots$ a set of generators of $\mathfrak{a}$. Following the proof of Theorem 5.1 we can express any reversible element w of $\{\mathfrak{a}\}$ as a Jordan polynomial in $x, y, u_1, u_2, \ldots$ and the tetrads in these variables. Moreover, this polynomial may be taken to be linear homogeneous in the u's, therefore no tetrads can occur, since now any tetrad must involve at least two distinct u's. Thus w can be written as a Jordan polynomial in $x, y, u_1, u_2, \ldots$ which is linear homogeneous in the u's and it follows that $w \in \mathfrak{a}$. Hence $\{\mathfrak{a}\} \cap J_0 \subseteq \mathfrak{a}$; this shows that $J_0/\mathfrak{a}$ is special and the proof is complete.

6. Example of a non-special $\mathfrak{T}$-algebra. In this section we construct an ideal of J_0 which does not satisfy the conditions of Theorem 5.1 and therefore defines a quotient algebra of J_0 which is not special.

We take the free generators of A_0 to be x, y, z and consider the element

$$u = \langle x, x\rangle - \langle y, y\rangle \qquad (= x^2 - y^2).$$

Let $\mathfrak{a}$ be the ideal of J_0 generated by u. We shall show that $(uxyz)^* \notin \mathfrak{a}$. Then $J_0/\mathfrak{a}$ will be non-special by Theorem 5.1.

Suppose that $w = (uxyz)^* \in \mathfrak{a}$. Then w can be written as a Jordan polynomial $\phi(u, x, y, z)$ and by an argument similar to that used in proving Lemma 3.2 we may suppose ϕ to be linear homogeneous in u. Let $\phi_n(u, x, y, z)$ be the sum of the terms of ϕ which are homogeneous of degree n in the last three arguments. Then

$$(uxyz)^* = \sum_n \phi_n(u, x, y, z).$$

Since A_0 is free, we may equate the homogeneous terms of degree n in x, y, z and this gives (because ϕ is linear homogeneous in u, and u is homogeneous in x, y, z)

$$(12) \qquad \begin{aligned} (uxyz)^* &= \phi_3(u, x, y, z), \\ \phi_n(u, x, y, z) &= 0 \qquad \text{for } n \neq 3. \end{aligned}$$

Now we decompose ϕ into parts which are homogeneous in each of its arguments: Let

$$\phi(u, x, y, z) = \sum \phi_{ijk}(u, x, y, z),$$

where ϕ_{ijk} is homogeneous of degree i, j, k respectively in the second, third and fourth argument. This process of picking out the homogeneous terms of a Jordan polynomial ϕ gives again a Jordan polynomial ϕ_{ijk} because a homogeneous Jordan polynomial, when considered as an associative polynomial, is still homogeneous.

We consider the homogeneous terms of degree 3, 1, 1 respectively in x, y, z in (12). Equating such terms, we get

$$(x^2yz)^* = \phi_{111}(x^2, x, y, z). \tag{13}$$

Similarly, by equating the terms of degree 1, 3, 1 in x, y, z we get

$$(y^2xyz)^* = \phi_{111}(y^2, x, y, z), \tag{14}$$

while ϕ_{ijk} with $(i, j, k) \neq (1, 1, 1)$ does not contribute to the result. Let us write

$$f(u, x, y, z) = (uxyz)^* - \phi_{111}(u, x, y, z),$$

where we regard u, x, y, z as independent. The associative polynomial f is linear homogeneous in each of its four arguments. Since ϕ_{111} is a Jordan polynomial, f is reversible; and it vanishes for $u = y^2 - x^2$, but does not vanish identically, because mod J_0 it is congruent to $(uxyz)^*$. From the vanishing of f for $u = x^2 - y^2$ we get

$$f(x^2, x, y, z) = f(y^2, x, y, z) = 0, \tag{15}$$

which is just another way of expressing the equations (13) and (14). If we treat u again as a fourth free variable, then $f(u, x, y, z)$, as a reversible linear homogeneous polynomial in u, x, y, z, must be a linear combination of the elements

$$\begin{aligned} &v_1 = (uxyz)^*, \ v_2 = (uxzy)^*, \ v_3 = (uyxz)^*, \ v_4 = (uyzx)^*, \\ &v_5 = (uzxy)^*, \ v_6 = (uzyx)^*, \ v_7 = (xuyz)^*, \ v_8 = (xuzy)^*, \\ &v_9 = (yuxz)^*, \ v_{10} = (yuzx)^*, \ v_{11} = (zuxy)^*, \ v_{12} = (zuyx)^*. \end{aligned}$$

For these twelve expressions form a basis for all the reversible elements which are linear homogeneous in u, x, y, z. Let $f(u, x, y, z) = \sum a_i v_i$, where the a's are in the underlying field. Then the equation $f(x^2, x, y, z) = 0$ implies that

$$a_1 + a_7 = a_2 + a_8 = a_9 + a_{11} = 0; \quad a_3 = a_4 = a_5 = a_6 = a_{10} = a_{12} = 0;$$

and $f(y^2, x, y, z) = 0$ implies that

$$a_3 + a_9 = a_4 + a_{10} = a_7 + a_{12} = 0; \quad a_1 = a_2 = a_5 = a_6 = a_8 = a_{11} = 0.$$

These equations together show that all the coefficients a_i vanish and hence $f(u, x, y, z)$ must vanish identically, which is a contradiction. Therefore $(uxyz)^* \notin \mathfrak{a}$ and so $J_0/\mathfrak{a}$ is not special. Expressing this result in terms of $\mathfrak{T}$-algebras we have

SCHOLIUM 6.1. *If J is the $\mathfrak{T}$-algebra[6] on three generators x, y, z with the single defining relation $x^2 = y^2$, then J is non-special, i.e., J cannot be embedded in an associative algebra.*

It is of interest to note that although the number of generators must be at least three (by Theorem 5.2), the defining relation involves only two of them. This means that the non-special $\mathfrak{T}$-algebra J of Scholium 6.1 is actually the

[6]The multiplication in J is here denoted by juxtaposition.

free product of two special Jordan algebras, namely the $\mathfrak{T}$-algebra on x and y with the single defining relation $x^2 = y^2$ and the free $\mathfrak{T}$-algebra on z.

The $\mathfrak{T}$-algebra J described in Scholium 6.1 also provides an example of a 3-generator $\mathfrak{T}$-algebra which cannot be embedded in a 2-generator $\mathfrak{T}$-algebra, for by Theorem 5.2 any such $\mathfrak{T}$-algebra can be embedded in an associative algebra A and this would provide an embedding of J in A.

7. $\mathfrak{T}$-algebras on more than four generators. The conditions of Theorem 5.1 can in principle be applied to any 3-generator $\mathfrak{T}$-algebra with a finite set of defining relations, but this may be a non-trivial problem for some $\mathfrak{T}$-algebras, and it is to be expected that any set of conditions for $\mathfrak{T}$-algebras on four or more generators will be equally if not more difficult to apply, and therefore less useful. Such conditions, if they exist, must be essentially different from the one given in Theorem 5.1. We shall briefly indicate the reason for this fact.

The criterion of Theorem 5.1 depends on the fact that for less than four generators the Jordan elements can be characterized by means of the reversal operator j and the crucial point is the application of Lemma 3.2 which states roughly that a reversible element of A_0 which is expressible as a polynomial of given degree in certain reversible elements of A_0 can also be expressed as a reversible polynomial of the same degree in these elements. More precisely, we can say that * is an idempotent operator permuting the places, such that, if $f(\xi_1, \xi_2, \ldots ; \eta_1, \eta_2, \ldots)\,(=f(\xi, \eta)$ for short) is any associative polynomial linear homogeneous in the η's, and f^* the polynomial defined by

$$f^*(\xi, \eta) = [f(\xi, \eta)]^*,$$

then

$$f(x, u)^* = f^*(x, u), \tag{16}$$

where the x's are free generators and the u's are reversible elements in the x's.

If there were a criterion similar to that of Theorem 5.1 for 4-generator $\mathfrak{T}$-algebras, with * replaced by a different idempotent operator permuting the places, or even by a series of such idempotent place-permutation operators, one for each set of homogeneous Jordan elements of a given degree,[7] then these operators would satisfy an analogue of Lemma 3.2 with $u_1, u_2, \ldots$ all homogeneous of the same degree in the free generators, and this would require (16) to hold for the new operators (with f homogeneous of the appropriate degree). From this it would follow that, for any Jordan element u homogeneous in x, y, z, the element

$$(uxyz)^*\,(=\tfrac{1}{2}(uxyz + zyxu))$$

satisfies the criterion, because qua polynomial in x, y, z it is a Jordan element by Theorem 4.2. Hence it could then be expressed as a Jordan polynomial in

[7] Such idempotent place-permutation operators exist in the case of Lie elements for homogeneous elements of any degree (cf. the remark after Lemma 3.1), but they do not give a characterization of Lie algebras which are embeddable in associative algebras because these operators do not satisfy the analogue of (16). We note however that Lie-algebras are embeddable in associative algebras in any case (cf. footnote 2).

u, x, y, z. But this contradicts the example constructed in §6, where $u = x^2 - y^2$. Therefore such idempotent place-permutation operators cannot exist.

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References

1. A. A. Albert, *On a certain algebra of quantum mechanics*, Ann. Math., *35* (1934), 65–73.
2. ———, *A structure theory for Jordan-algebras*, Ann. Math., *48* (1947), 546–567.
3. H. F. Baker, *Alternants and continuous groups*, Proc. Lond. Math. Soc. (2), *3* (1905), 24–47.
4. G. Birkhoff, *Representability of Lie-algebras and Lie-groups by matrices*, Ann. Math., *38* (1937), 526–532.
5. ———, *Universal algebra*, Proc. Can. Math. Congress (Montreal 1946), 310–326.
6. G. Birkhoff and P. M. Whitman, *Representation of Jordan and Lie algebras*, Trans. Am. Math. Soc., *65* (1949), 116–136.
7. F. D. Jacobson and N. Jacobson, *Classification and representation of semisimple Jordan algebras*, Trans. Am. Math. Soc., *65* (1949), 141–169.
8. N. Jacobson and C. E. Rickart, *Jordan homomorphisms of rings*, Trans. Am. Math. Soc., *69* (1950), 479–502.
9. F. Wever, *Invarianten in Lieschen Ringen*, Math. Ann., *120* (1949), 563–580.
10. E. Witt, *Treue Darstellung Liescher Ringe*, J. reine angew. Math., *177* (1937), 152–160.

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STABLE LATTICES

PART II

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10. Introduction to Part II. The most interesting cases of stable lattices, introduced in an earlier volume of this journal **(12)**, were the (algebraic) modules of stable norm, or modules whose ratio of minimum absolute non-zero norm to lattice determinant (i.e., to the square root of module-discriminant) is a local maximum for small variations of the basis. We soon found that these modules were perhaps more numerous than we should have desired if we were interested only in finding an absolute maximum. Nevertheless stable lattices acquire a certain amount of intrinsic interest once we reach the stage where the *continuous* variable concept of "neighborhood of a lattice" leads to *algebraic* criteria of stability. In both these respects the situation displays some similarity to the more classical subject of extremal (quadratic) forms **(2)**.

If the module in question is the module of all integers (i.e., the so-called *integer-module*) of a *totally real* field of degree n, then the criterion of stability depends on the relative signs of the conjugates of the units. This is a specialization, (*weaker* when $n > 3$), of the classical concept of "(maximum) signature rank 2^n," used by Weber **(13)** for instance. We shall merely show, as a significant illustration, that the integer-module of the totally real field $K_N = R(\cos 2\pi/N)$, over the rationals R, has stable norm (except for a few small N).

If we consider fields that are *not totally real*, the classical problem of signature rank becomes vacuous for the imaginary (conjugate) fields while the stability criterion that we use becomes more complicated. We shall, however, demonstrate a refinement of the "unit star" method of Part I **(12**; p. 265**)**, to obtain a criterion that can be readily used to test more general modules. We shall carry this through the cubic case.

The big difficulty arises when "too many" roots of unity occur. We shall, therefore, consider as a final example the integer-module of the cyclotomic field $K^N = R(\exp 2\pi i/N)$. This module turns out to have a stable norm if and only if N is square-free. This result, which essentially uses the presence or absence of a normal basis for K^N, should further attest to the algebraic aspect of stability.

11. A totally-real illustration. If we specialize somewhat to integer-modules of a totally real *normal* field K, then the criterion of stability (of the

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norm), given in Part I **(12**, p. 269**)**, reduces to the following: *For any non-identical automorphism S of K, (over the rationals), at least one unit u exists for which u/u^S is negative.*

For instance, consider the totally real field $K_N = R(\cos 2\pi/N)$ of degree $n = \frac{1}{2}\phi(N)$ (when $N > 2$). Here $\phi(N)$ is the Euler totient function (not to be confused with the gauge function of Part I **(12**, p. 261**)**, which has a vector argument). The field K_N is the maximal totally real field in the cyclotomic field $K^N = R(\exp 2\pi i/N)$. We shall show that *the integer-module of K_N has stable norm if*

$$N \neq 1, 2, 3, 4, 6, 12. \tag{11.1}$$

For the first five exceptional values of N, K_N is rational; while for $N = 12$, $K_{12} = R(\sqrt{3})$, which is unstable by the unsolvability **(12**, p. 267**)** of the diophantine equation $x^2 - 3y^2 = -1$.

For the field K_N, the Galois group consists of the $\frac{1}{2}\phi(N)$ operations

$$S: \zeta \to \zeta^{\pm g}, \qquad (g, N) = 1. \tag{11.2}$$

$$(\cos 2\pi M/N)^S = \cos 2\pi Mg/N. \tag{11.3}$$

To prove stability we make use of the following types of units:

$$\theta = 2\cos 2\pi M/N \qquad N \neq 2^k,\ 4p^k\ (k > 0), \tag{11.4}$$

$$\psi = 1 + 2\cos 2\pi M/N \qquad N \neq 3p^k \quad (k > 0), \tag{11.5}$$

where $(M, N) = 1$ and p is a prime. (We note that this choice of units of K_N accounts for all N except the values (11.1).)

Now first of all we assert that for the special values of N:

$$N_1 = p^k (\neq 3), \quad N_2 = pq \tag{11.41}$$

the units θ are sufficient to establish stability; while for the special values:

$$N_3 = 4.2^k, \quad N_4 = 4p (\neq 12) \tag{11.51}$$

the units ψ are sufficient. In these cases p and q are odd primes and $k > 1$. We shall verify this just in a typical case, $N = N_1$. Here all we need to show is that for a given non-trivial automorphism S, a $M = M_1$ exists for which $\theta = \theta_1$ is positive and θ_1^S is negative. Thus all we need show is that for any $g \not\equiv \pm 1 \bmod N_1$, (for which $(g, N_1) = 1$), there exists an M_1 (for which $(M_1, N_1) = 1$), such that M_1 lies in the range $(-\frac{1}{4}N_1, \frac{1}{4}N_1)$, while $M_1 g$ is congruent (mod N_1) to a value in the range $(\frac{1}{4}N_1, \frac{3}{4}N_1)$. The existence of such an M_1 is assured from the fact that the possible values of M (i.e., the non-multiples of p), have such few gaps (i.e., the multiples of p), that the values of Mg cannot always "straddle" the range $(\frac{1}{4}N_1, \frac{3}{4}N_1)$ modulo N_1. Thus some value of Mg must lie in this range when $M = M_1$, a non-multiple of p. A similar argument holds for N_2, N_3, and N_4 (which were singled out for simplicity of proof, since they have few prime divisors and hence "small gaps").

We are now prepared to handle the general N in the following way: Suppose

that for the given S, $g \not\equiv \pm 1 \bmod N_1$, for N_1 (of type (11.41)) dividing N. Then the very same θ_1 used earlier is a unit of K_{N_1} (hence of K_N) for which θ_1/θ_1^S is negative. Thus we see that the difficult case becomes the one in which, for each $p_i^{k_i}$ in the factorization $N = \Pi p_i^{k_i}$ the congruence

$$g \not\equiv \pm 1 \bmod p_i^{k_i}$$

is satisfied. In order that S still not be the identity, it is necessary that these ± 1 be "mixed" (e.g., that g not be congruent to (say) $+1$ for each $p_i^{k_i}$. Thus $g \not\equiv \pm 1$ for some composite modulus dividing N, of the type N_2 (see (11.41)) or N_4 (see (11.51)). Hence, once more, as in the preceding paragraph, a unit θ_2 or ψ_4 is determined for which θ_2/θ_2^S or ψ_4/ψ_4^S is negative. Q.E.D.

12. The stability configuration. In the last section, we obtained only an existence proof, i.e. a proof that $Q + 1$ $(= n(n-1)+1)$ *positively dependent* "gradient" vectors **(12**, pp. 263, 266**)**.

$$\mathbf{R}[\mathbf{u}] = (\ldots, u_l/u_j, \ldots) \qquad (l, j = 1, 2, \ldots, n; l \neq j), \tag{12.1}$$

exist, where $\mathbf{u} = (u_1, \ldots, u_n)$ is a vector of the module formed from each of $Q + 1$ properly chosen units u_l of K_N. The set of $Q + 1$ vectors $\mathbf{R}[\mathbf{u}]$ will be called a *stability configuration*. The actual choice of the $Q + 1$ units u_l, as we shall indicate, is not only extremely difficult, but it accentuates irregularities in the infinitesimal behavior of the units of K_N for different N.

Nevertheless the finding of these $Q + 1$ vectors $\mathbf{u}$ or $\mathbf{R}[\mathbf{u}]$ would be desirable in view of the fact that a modular reduction theorem automatically enlarges the "neighborhood" of a stable lattice (*possibly* to the whole lattice space, thus establishing a critical lattice, as in the quadratic case **(12**, p. 268**)**). For instance, take the following simple-looking set of n units of K_N (for N prime $\geqslant 5$):

$$u_1 = \begin{cases} \zeta & + \zeta^{-1} \\ \zeta^3 & + \zeta + \zeta^{-1} + \zeta^{-3} \\ \ldots & \qquad\qquad \ldots \\ \zeta^{N-4} + \zeta^{N-6} + \ldots + \zeta + \zeta^{-1} + \ldots + \zeta^{-N+6} & + \zeta^{-N+4} \\ -1 \end{cases} \tag{12.2}$$

where $\zeta = \exp 2\pi i/N$. Each of the first $n - 1$ units listed above is taken with its conjugates and the last (rational) unit is taken once, forming $Q + 1 = n(n-1) + 1$ units.

It is found by very laborious calculations (which we omit) that the resulting $Q + 1$ vectors $\mathbf{R}[\mathbf{u}]$ provide a stability configuration when $N = 5, 7, 13$ but that these vectors are not even of rank Q when $N = 11$. The question of whether or not the integer-module of K_N provides a critical lattice for the norm in n real dimensions is answered positively for $N = 5, 7$ **(3)**, and negatively for $N = 13$ **(11)**, while for $N = 11$ the answer is still unknown. It would probably be wise to exhibit a sufficiently simple stability configuration for the integer-module in K_{11} before trying to establish it as a critical lattice.

13. Stability criterion for complex modules. Going to the complex case, we return to greater generality by considering as in §9 (above) any (non-degenerate) module $\mathfrak{M}$ (not necessarily consisting of *all* the integers of a field). The elements of $\mathfrak{M}$ are algebraic integers z with r real and $2s$ complex conjugates:

$$\begin{aligned} z_j &= x_j && (j = 1, 2, \ldots, r), \\ z_j &= x_j + i\,x_{j+s} && (j = r+1, \ldots, r+s), \\ z_{j+s} &= x_j - i\,x_{j+s}, \end{aligned} \tag{13.1}$$

making a total of $r + 2s = n$ conjugates. Then we are interested in stable modules for the norm (gauge) function

$$|\,x_1 \ldots x_r\,(x_{r+1}^2 + x_{r+s}^2) \ldots (x_{r+s+1}^2 + x_{r+2s}^2)|.$$

We consider the vector of $Q = n(n-1)$ generally complex components:

$$\mathbf{R}[\mathbf{z}] = (\ldots, z_l/z_j, \ldots) \quad (l, j = 1, 2, \ldots, n; l \neq j). \tag{13.2}$$

Then the criterion for stability is that the set of vectors $\mathbf{R}[\mathbf{z}]$ positively span a space of dimension Q as $\mathbf{z}$ varies over the set of module elements of minimum non-zero absolute norm. This criterion was established earlier (**12**, p. 266), using $2Q$ real components, but the present form will prove simpler to use. As in the totally real case, the finding of $Q+1$ positively dependent vectors $\mathbf{R}[\mathbf{z}]$ is so difficult that we must establish a new criterion relating to *projections of* $\mathbf{R}[\mathbf{z}]$.

Now every vector $\mathbf{R}[\mathbf{z}]$ can be expressed in terms of $n(n-1)$ *real* coordinates if we break the vector up into the following projections: Take any pair of distinct fields chosen from the first $r+s$ conjugate fields (conjugate complex repetitions being excluded). For every such pair, denoted by unequal subscripts $l, j (\leqslant r+s)$, a projection is determined. Specifically for l, j *both real*, there are $r(r-1)$ projections

$$\mathbf{R}_{l,j}[\mathbf{z}] = (z_l/z_j), \tag{13.31}$$

for l, j *both complex*, there are $s(s-1)$ projections

$$\mathbf{R}_{l,j}[\mathbf{z}] = (z_l/z_j,\ z_{l+s}/z_j) \tag{13.32}$$

for l, j *mixed* (i.e., one real and the other complex), there are $2rs$ projections

$$\mathbf{R}_{l,j}[\mathbf{z}] = (z_l/z_j). \tag{13.33}$$

Now, clearly, the projections of type (13.31) are of real dimension one; the projections of type (13.32) and (13.33) having two and one *complex* components, are of real dimension four and two respectively. We write this as dim $[l, j] = 1, 4$, and 2 respectively. If we were to sum the number of coordinates, we should find it accounts for

$$r(r-1) \cdot 1 + s(s-1) \cdot 4 + 2rs \cdot 2 = Q - 2s$$

coordinates. The remaining $2s$ coordinates are accounted for by a single projection called the "$2s$" projection,

$$\mathbf{R}_{2s}[\mathbf{z}] = (z_{r+1}/z_{r+s+1}, \ldots, z_{r+s}/z_{r+2s}), \tag{13.34}$$

of s complex components and of (real) dimension $2s$.

For some purposes it will be necessary to visualize all the real and imaginary parts written out as Q cartesian coordinates of a cartesian space $\mathfrak{S}^Q$ with projections $\mathfrak{S}^{\dim[l,j]}$ and $\mathfrak{S}^{2s}$. Thus we may introduce a (real) scalar product and with it length and angle. Furthermore any one of the projections may be imbedded (in the natural way) in $\mathfrak{S}^Q$ by setting the $Q - \dim[l, j]$ or $Q - 2s$ remaining coordinates equal to zero. Thus we may speak of the angle between $\mathbf{R}[\mathbf{z}]$ and any one of its projections, $\mathbf{R}_{l,j}[\mathbf{z}]$ or $\mathbf{R}_{2s}[\mathbf{z}]$.

The significance of the subspaces is then derived from the following fact: *Let any* $\mathbf{z}_0$ $(\neq 0)$ *be given. Then a* $\mathbf{z}_1$ *of equal norm exists for which* $\mathbf{R}[\mathbf{z}_1]$ *is arbitrarily close in direction to any preassigned* $[l, j]$ *projection* $\mathbf{R}_{l,j}[\mathbf{z}_0]$. To see this, take $\mathbf{z}_1 = \mathbf{z}_0\mathbf{u}$ where $\mathbf{u}$ is a unit of $\mathfrak{O}$ the order of $\mathfrak{M}$, so chosen (by Dirichlet's Theorem on units **(12**, p. 269**)**), that $|u_p/u_q|$ has the largest order of magnitude for the choice of sub-scripts $p = l$, $q = j$ $(p, q \leqslant r + s)$, and so that u_l/u_j lies arbitrarily close in argument to a *positive* real number. The main stability criterion will be a more complicated version of the "unit star" method of Part I **(12**, p. 265**)**, (which dealt entirely with one-dimensional $[l, j]$ projections). It is as follows:

The necessary and sufficient condition that the module $\mathfrak{M}$ *have a stable norm is that for* $\mathbf{z}$ *a variable element of minimum absolute non-zero norm in* $\mathfrak{M}$, *the projections* $\mathbf{R}_{l,j}[\mathbf{z}]$ *or* $\mathbf{R}_{2s}[\mathbf{z}]$ *each positively span a space of* $\dim[l, j]$ *or* $2s$ *dimensions.*

The *necessity* is immediate since a projection of a set of vectors positively spanning a space must necessarily positively span the projection. The *sufficiency* proof is the difficult one.

14. Sufficiency proof of stability criterion. We let $\mathbf{z}$ be a general element of $\mathfrak{M}$ of minimum absolute non-zero norm and we assume, as our hypothesis, that for certain values of $\mathbf{z}$, namely

$$\mathbf{z}_{l,j}^{(h)}, \mathbf{z}_{2s}^{(k)} \qquad (1 \leqslant h \leqslant 1 + \dim[l, j], \quad 1 \leqslant k \leqslant 1 + 2s), \tag{14.1}$$

the sets of *projection* vectors $\{\mathbf{v}\}$, consisting of the subsets

$$\begin{aligned} &\mathbf{R}_{l,j}[\mathbf{z}_{l,j}^{(h)}] = \{\mathbf{v}\}_{l,j} && (1 \leqslant h \leqslant 1 + \dim[l, j]), \\ &\mathbf{R}_{2s}[\mathbf{z}_{2s}^{(k)}] = \{\mathbf{v}\}_{2s} && (1 \leqslant k \leqslant 1 + 2s), \end{aligned} \tag{14.2}$$

are positively dependent. The object is to show that for properly chosen units $\{\mathbf{U}\}$ of the order $\mathfrak{O}$ the vectors $\{\mathbf{W}\}$, consisting of the subsets

$$\begin{aligned} &\mathbf{R}[\mathbf{Z}_{l,j}^{(h)}] = \{\mathbf{W}\}_{l,j} && (1 \leqslant h \leqslant 1 + \dim[l, j]), \\ &\mathbf{R}[\mathbf{Z}_{2s}^{(k)}] = \{\mathbf{W}\}_{2s} && (1 \leqslant k \leqslant 1 + 2s), \end{aligned} \tag{14.3}$$

positively span the real cartesian space $\mathfrak{S}^Q$, where

$$\mathbf{Z}_{l,j}^{(h)} = \mathbf{z}_{l,j}^{(h)}\mathbf{U}_{l,j}^{(h)}, \quad \mathbf{Z}_{2s}^{(k)} = \mathbf{z}_{2s}^{(k)}\mathbf{U}_{2s}^{(k)}. \tag{14.4}$$

The positive span will be established by means of the *projection* and *independence*

properties (12, p. 265). We use the further set $\{\mathbf{V}\}$ to represent the projections of $\{\mathbf{W}\}$:

$$\mathbf{R}_{l,j}[\mathbf{Z}_{l,j}^{(h)}] = \{\mathbf{V}\}_{l,j} \qquad (1 \leqslant h \leqslant 1 + \dim[l,j]),$$
$$\mathbf{R}_{2s}[\mathbf{Z}_{2s}^{(k)}] = \{\mathbf{V}\}_{2s} \qquad (1 \leqslant k \leqslant 1 + 2s). \tag{14.5}$$

The arguments used will be quite general (e.g., the values of the dimensionalities are of no importance).

First of all choose the $\mathbf{U}_{2s}^{(k)}$ $(1 \leqslant k \leqslant 1 + 2s)$ so that the $2s + 1$ projection vectors of the set $\{\mathbf{V}\}_{2s}$ are (still) positively dependent (in $\mathfrak{S}^{2s}$) while the vectors $\{\mathbf{W}\}_{2s}$ are independent (in the larger space $\mathfrak{S}^{Q}$). This can be done by the approximation technique of §13 (above), since the proper choice of the $\mathbf{U}_{2s}^{(k)}$ for each k will render each vector of $\{\mathbf{v}\}_{2s}$ arbitrarily close in direction to the corresponding vector in $\{\mathbf{V}\}_{2s}$ while some one $[l, j]$ projection of one vector of $\{\mathbf{W}\}_{2s}$ can be made much larger than this projection of all the other vectors of $\{\mathbf{W}\}_{2s}$. By this double condition on the $\mathbf{U}_{2s}^{(k)}$, $(1 \leqslant k \leqslant 1 + 2s)$, we preserve the positive dependence of the set $\{\mathbf{V}\}_{2s}$ in $\mathfrak{S}^{2s}$ without having the positive dependence relation remain valid for the set $\{\mathbf{W}\}_{2s}$ in $\mathfrak{S}^{Q}$. Once these $\mathbf{U}_{2s}^{(k)}$ are chosen we keep them fixed for the remainder of the proof.

Next we assign a positive angle ϵ. Then for this ϵ we can choose units $\mathbf{U}_{l,j}^{(h)}$, $(1 \leqslant h \leqslant 1 + \dim[l,j])$, such that $\{\mathbf{V}\}_{l,j}$ are still positively dependent (for each $[l, j]$) in the smaller space $\mathfrak{S}^{\dim[l,j]}$ while $\{\mathbf{W}\}_{l,j}$ are linearly independent in the larger space $\mathfrak{S}^{Q}$. *At the same time*, each vector of the set $\{\mathbf{W}\}_{l,j}$ is to make an angle $< \epsilon$ with the corresponding vector of the set $\{\mathbf{V}\}_{l,j}$. This can be done, as in the preceding paragraph by the approximation technique of §13 (above).

We now show that for ϵ small enough the *projection* property is valid, or that any arbitrary (say) *unit* vector $\mathbf{T}$ of $\mathfrak{S}^{Q}$ has a positive projection on one or more vectors of the total set $\{\mathbf{W}\}$. For if $\mathbf{T}$ lies wholly in the $\mathfrak{S}^{2s}$ projection, the positive dependence of $\{\mathbf{V}\}_{2s}$ is sufficient. On the other hand if $\mathbf{T}$ has a non-zero projection in some other space $\mathfrak{S}^{\dim[l,j]}$ then for ϵ small enough some element of $\{\mathbf{W}\}_{l,j}$ (being sufficiently close in direction to an element of $\{\mathbf{V}\}_{l,j}$), must have a positive projection on $\mathbf{T}$. By compactness, a single ϵ should achieve this uniformly for all directions of $\mathbf{T}$.

We next show that for ϵ small enough, the *independence* property is valid, or any Q vectors of the set $\{\mathbf{W}\}$ are linearly independent. Consider first the projection vectors $\{\mathbf{V}\}$. Clearly any linear relationship among vectors in $\{\mathbf{V}\}$ must be decomposable into the sum of linear relationships among vectors of the individual sets $\{\mathbf{V}\}_{l,j}$ and $\{\mathbf{V}\}_{2s}$ (lying wholly in the corresponding projections $\mathfrak{S}^{\dim[l,j]}$ and $\mathfrak{S}^{2s}$). Now by continuity this will be true for vectors sufficiently close in direction to the vectors of $\{\mathbf{V}\}$. The $\{\mathbf{W}\}_{l,j}$ immediately meet this condition when ϵ is small enough (by our choice of the units $\mathbf{U}$). The $\{\mathbf{W}\}_{2s}$ will also meet this condition if at the same time that we make ϵ small we stretch each of the $2s$ coordinates in $\mathfrak{S}^{2s}$ to infinity sufficiently *slowly* so as not to ruin the approximation of $\{\mathbf{W}\}_{l,j}$ to $\{\mathbf{V}\}_{l,j}$. This will accentuate the $\mathfrak{S}^{2s}$ projection of the *fixed* vectors $\{\mathbf{W}\}_{2s}$ and (effectively) permit them to approximate $\{\mathbf{V}\}_{2s}$.

Thus any linear relationship among vectors of the set $\{\mathbf{W}\}$ will imply such relationships among vectors in the individual sets $\{\mathbf{W}\}_{2s}$ or $\{\mathbf{W}\}_{l,j}$ which, as we saw earlier, are trivial, by choice of $\{\mathbf{U}\}$, for ϵ small enough. Q.E.D.

Now the existence of $Q+1$ positively dependent vectors, chosen from the $\sum(1+\dim[l,j])+(2s+1)$ vectors $\{\mathbf{W}\}$, is assured by a general convexity argument **(12**, p. 265**)**, but in practice the difficulties in exhibiting these $Q+1$ vectors are, as in the real case, almost prohibitive. (Regrettably, our earlier bibliography failed to list the work of Carathéodory **(10)** and Steinitz **(15)** who seem to have used similar convexity arguments, in a different context. The method of approximating projections, so natural with Dirichlet's theory of units **(5)**, seems to have not been used previously.)

15. Quadratic and cubic modules. The question of stability in a totally real module has already been answered in Part I **(12**, p. 269**)**. For complex modules the answer becomes extremely involved as the degree n increases. We carry the investigation only as far as $n=3$.

Starting with *quadratic* (complex) modules, we find that stability requires first of all that the norm assume its minimum at three $(=Q+1)$ values (and their negatives), lying on a circle with center at the origin. This shows the lattice to be equivalent under rotation to the equilateral lattice. Thus the *only stable quadratic (complex) modules are given by the module* $\rho\mathfrak{O}$, *where* ρ *is a fixed integer and* $\mathfrak{O}$ *is the set of all integers, in* $R(\exp 2\pi i/3)$.

In the case of cubic modules with one real and two complex conjugates $(n=3, r=1, s=1)$, *we find*, as indicated earlier without proof **(12** p. 269**)**, *that all such modules have stable norms*. To see this we start with the observation that the order $\mathfrak{O}$ of $\mathfrak{M}$ has one fundamental unit $\mathbf{w}=(w_1, w_2, w_3)$. Now w_2 is not real nor is any power of w_2 real, since the only real numbers in a complex cubic field are rational. This shows that the argument of w_2 is incommensurable with 2π, and therefore the directions of the complex vectors w_2^m, $(m=0, \pm1, \pm2, \ldots)$ are everywhere dense mod 2π. Thus for every $\mathbf{z}_0$ of minimal norm, the set $\mathbf{z}=\mathbf{z}_0\mathbf{w}^s$ has the same norm while the projections $\mathbf{R}_{1,2}[\mathbf{z}]$, $\mathbf{R}_{2,1}[\mathbf{z}]$, $\mathbf{R}_{2s}[\mathbf{z}]$, each in two-dimensional space, are everywhere dense with respect to direction, thus establishing the positive span and stability. (For an illustrative stability configuration see **(7)**.)

When $n>3$ the criteria are dominated by the occurrence of units and combinations of units whose arguments are commensurable with 2π. Here, the subject of the geometry of numbers must await more results in algebra. We shall, however, give such a case as the final illustration.

16. Cyclotomic field. We conclude by proving the *stability of the integer module in* $K^N=R(\exp 2\pi i/N)$ *if and only if* N *is square-free*. Here we note that the arguments of all units u are commensurable with 2π **(14**, p. 334**)**. In fact, denoting the conjugate-complex of u by $\bar{u}$, we find $u/\bar{u}$ is always a root of unity

lying in K^N hence a positive or negative power of $(\zeta =) \exp 2\pi i/N$. In this section $\phi(N) = n$, the degree of K^N over R.

First we dispose of the case where N has a square factor, say p^2, for p a prime. Consider any $n+1$ $(=2s+1)$ projection vectors $\mathbf{R}_{2s}[\mathbf{u}]$ for $n+1$ units $\mathbf{u}$; we shall show that such a set of vectors can never be positively dependent. For the s complex coordinates of $\mathbf{R}_{2s}[\mathbf{u}]$ (see equation (13.34)) are the conjugates of some positive or negative power of ζ. Thus if we write out any $n+1$ powers of ζ we find that some $(n/p)+1$ of them belong to the same residue class module p and are therefore linearly dependent (with rational coefficients), by virtue of the fact that the cyclotomic equation of degree n, defining ζ, is really an equation of degree n/p in ζ^p. Hence, by conjugates, a linear (rational) dependence exists among a proper subset of any $n+1$ vectors $\mathbf{R}_{2s}[\mathbf{u}]$, excluding positive dependence.

From now on, we assume N is square-free, hence necessarily odd (and greater than 3 for convenience). We build the positive dependence from the relation

$$\sum_{k=1}^{n} \zeta^{a_k} - \mu(N) = 0 \tag{16.1}$$

where the a_k are the $n(=\phi(N))$ residue classes relatively prime to N, and $\mu(N)$ is the Moebius inversion function. We then take the $2s+1$ units (of which one conjugate is indicated):

$$\begin{aligned} u_1^{(k)} &= \zeta^{\frac{1}{2}a_k(N+1)} && (k = 1, 2, \ldots, n), \\ u_1^{(n+1)} &= 1 && \text{if } \mu(N) = -1, \\ &= (1-\zeta)^N && \text{if } \mu(N) = +1, \end{aligned} \tag{16.2}$$

(since $1-\zeta$ is a unit now if and only if N is no prime power). We further see that

$$\begin{aligned} u_1^{(k)}/\bar{u}_1^{(k)} &= \zeta^{a_k} && (k = 1, 2, \ldots, n), \\ u_1^{(n+1)}/\bar{u}_1^{(n+1)} &= -\mu(N), \end{aligned} \tag{16.3}$$

and hence from (16.1)

$$\sum_{k=1}^{n+1} \mathbf{R}_{2s}[\mathbf{u}^{(k)}] = 0. \tag{16.4}$$

Now equation (16.4) asserts positive span in the "$2s$" projection once we know that *all* linear relations among $\mathbf{R}_{2s}[\mathbf{u}^{(k)}]$ are proportional to the relation (16.4). Otherwise expressed, we assert there is only a trivial relationship among the $\mathbf{R}_{2s}[\mathbf{u}^{(k)}]$ if k takes the values $1 \leqslant k \leqslant n$. Assume, to the contrary, that the system

$$\sum_{k=1}^{n} A_k \zeta^{a_k a_j} = 0 \qquad 1 \leqslant k,\ j \leqslant n, \tag{16.5}$$

has a real non-trivial solution in A_k. Then a relation must exist where the A_k belong to K^N and (say) $A_1 = 1$. By applying the Galois group operations to (16.5) we derive a system in which the A_k are rational and not all zero. Thus

the relationship (16.5) contradicts the fact that ζ and its conjugates ζ^{a_k} ($k = 1, 2, \ldots, n$) provide a (normal) basis of K^N. (This last result is best known **(14**, p. 351**)** when N is prime; when N is composite but square-free, the direct-product field decomposition extends the result).

Having disposed of the "$2s$" projection, we must show that the $[l, j]$ projections, (see equation (13.32) above), are positively spanned. As in the beginning of §11 (above), we can reduce the problem to the following:

Let S be any non-identical automorphism of K^N over the rationals, distinct from $i \to -i$, then five units $u^{(h)}$ $(1 \leqslant h \leqslant 5)$ can be found for which the five vectors $\mathbf{R}_S{}^{(h)} = (u^{(h)}/u^{(h)S}, \bar{u}^{(h)}/u^{(h)S})$ are positively dependent. Here S is given by the transformation:

$$S\colon \zeta \to \zeta^g \qquad (g, N) = 1,\ g \not\equiv \pm 1 \pmod{N}. \tag{16.6}$$

Then we can take as the five units $u^{(h)} = \zeta, \zeta^{-1}, \zeta^a, \zeta^{-a}, 1$ respectively, where a is chosen as a function of g in such a manner that $\mathbf{R}_S{}^{(h)}$ are positively dependent. The specific details are tedious and are omitted as they have no bearing on the square-free nature of N but rather depend on the "small gap" type of argument used to determine M_1 in §11.

References

10. C. Carathéodory, *Über den Variabilitätsbereich der Fourier'schen Konstanten*, Rend. Circ. Math. Palermo, *32* (1911), 216.

11. H. Cohn, *Note on fields of small discriminant*, Proc. Amer. Math. Soc., *3* (1952), 713–714.

12. ———, *Stable lattices*, Can. J. Math., *5* (1953), 261–270.

13. H. Hasse, *Über die Klassenzahl abelscher Zahlkörper* (Berlin, 1952), 26–30.

14. D. Hilbert, *Theorie der algebraischen Zahlkörper*, Jahresbericht der deutschen Mathematiker-Vereinigung, *4* (1894), 177–546.

15. E. Steinitz, *Bedingt konvergente Reihen und konvexe Systeme*, J. reine angew. Math., *144* (1914), 15.

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FINITE UNITARY REFLECTION GROUPS

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Introduction. Any finite group of linear transformations on n variables leaves invariant a positive definite Hermitian form, and can therefore be expressed, after a suitable change of variables, as a group of unitary transformations **(5**, p. 257**)**. Such a group may be thought of as a group of congruent transformations, keeping the origin fixed, in a unitary space U_n of n dimensions, in which the points are specified by complex vectors with n components, and the distance between two points is the norm of the difference between their corresponding vectors.

In the real case we have a group of orthogonal transformations in Euclidean space E_n. Among such groups the *groups generated by reflections* have been the object of considerable study **(6**; **9**; **10)**. The concept of a reflection has recently been extended to unitary space by Shephard **(24)**. A reflection in unitary space is a congruent transformation of finite period that leaves invariant every point of a certain prime, and it is characterised by the property that all but one of the characteristic roots of the matrix of transformation are equal to unity. The remaining root, if the reflection is of period m, is a primitive mth root of unity, and the reflection is then said to be m-fold. Shephard, in the paper just quoted, has considered a particular class of unitary groups generated by reflections which possess properties closely analogous to those of the real orthogonal groups considered by Coxeter.

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This paper is divided into two parts. In the first part we determine all finite *unitary groups generated by reflections*. To save constant repetition we shall abbreviate the italicised words to 'u.g.g.r.', the qualification 'finite' being understood throughout. Since any finite group of unitary transformations is either irreducible or completely reducible **(5**, p. 263**)**, it suffices to determine the irreducible groups; in fact, if a u.g.g.r. is reducible, it is necessarily the direct product of its irreducible components, each of which is itself a u.g.g.r. This is accomplished in §§1–4. The method employed depends on the fact that, with each u.g.g.r. $\mathfrak{G}$ in U_n we can associate a collineation group $\mathfrak{G}'$ in projective space of $n-1$ dimensions which is generated by homologies, and that conversely, from each such group $\mathfrak{G}'$ a finite number (generally only one) of u.g.g.r. can be derived. We are thus able to draw on the considerable literature which exists on collineation groups generated by homologies. In fact all the collineation groups we require are well known except the imprimitive ones, and these, as we shall see, can be determined without difficulty.

In the second part of the paper we place on record some curious properties of these groups, which are enunciated in the form of theorems in §5. These extend and generalise the results obtained by Coxeter **(10)** for the special type of u.g.g.r. which consists of real orthogonal transformations. The next five sections of the paper are devoted entirely to the proofs of these theorems. The last section gives abstract definitions for those groups in U_n which are generated by n reflections.

I. DETERMINATION OF ALL THE FINITE IRREDUCIBLE UNITARY GROUPS GENERATED BY REFLECTIONS

1. The associated collineation group $\mathfrak{G}'$. Let $\mathfrak{G}$ be a u.g.g.r. in U_n. The matrices which correspond to the operations of $\mathfrak{G}$ can be regarded as collineation matrices in projective space S_{n-1} of $n-1$ dimensions, and the corresponding collineations form a group $\mathfrak{G}'$ which is isomorphic to the quotient group of $\mathfrak{G}$ by the cyclic subgroup $\mathfrak{Z}$ which consists of the elements of $\mathfrak{G}$ represented by scalar matrices. To the m-fold reflections of $\mathfrak{G}$ correspond collineations of finite period m leaving fixed all points of a prime of S_{n-1}. Such collineations are known as *homologies*, and thus $\mathfrak{G}'$ is generated by homologies.

Conversely, suppose $\mathfrak{G}'$ is a collineation group in S_{n-1} generated by homologies. The matrix of any such homology has a set of characteristic roots of the form $(\lambda, \lambda, \ldots, \lambda, \mu)$ with $\lambda \neq \mu$, $\lambda\mu \neq 0$. If $n > 2$ this matrix can be normalised in a unique way (by multiplying by λ^{-1}) so that it has all but one of its characteristic roots equal to unity, while if $n = 2$ this can be done in two ways (the other multiplier being μ^{-1}); and such a normalised matrix is equivalent to a reflection. Thus from any group $\mathfrak{G}'$ generated by homologies we can construct, in at most a finite number of ways, a group $\mathfrak{G}$ generated by reflections. More precisely, this can be done in a unique way provided that $n > 2$ and that the cyclic subgroups generated by the homologies in $\mathfrak{G}'$ are all conjugate in $\mathfrak{G}'$. As we shall soon see, these subgroups are conjugate when $n > 2$ except in a very few cases. But when

$n = 2$ a variety of cases arise which need separate discussion. When $n = 1$, $\mathfrak{G}$ is evidently a cyclic group generated by a reflection, and $\mathfrak{G}'$ is trivial.

2. Imprimitive u.g.g.r. in U_n. The groups $\mathfrak{G}$ and $\mathfrak{G}'$ are clearly both primitive or both imprimitive. In this section we shall determine the irreducible u.g.g.r. in U_n which are imprimitive.

For any imprimitive group $\mathfrak{G}$ it must be possible to arrange the transformation variables in sets which are either unchanged or permuted among themselves by every operation of $\mathfrak{G}$. If $\mathfrak{G}$ is irreducible, these sets of variables must be permuted transitively, so that each set contains the same number, n_1 say, of variables. The number of sets is then k where $kn_1 = n$. If we consider the corresponding collineation group $\mathfrak{G}'$ in S_{n-1}, there are then k linear spaces of dimension $n_1 - 1$, whose join is S_{n-1}, which are permuted among themselves by the collineations of $\mathfrak{G}'$. Now $\mathfrak{G}'$ is generated by homologies, and in any homology, a space of dimension $r > 0$ which is not invariant under the homology meets its transform in a space of dimension $r - 1$. It follows that, for an irreducible imprimitive group generated by homologies in S_{n-1}, we must have $n_1 = 1$, $k = n$, and therefore there are n distinct isolated points which are permuted by the collineations of $\mathfrak{G}'$. These n points define a simplex (which we naturally take as the simplex of reference) and the group $\mathfrak{G}'$ then consists of monomial substitutions of the form:

$$2.1 \qquad T: \quad x_i' = a_i x_{\sigma(i)}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation σ of $(1, 2, \ldots, n)$. If all the characteristic roots of this, save one, are equal to unity, the transformation must be of one or other of the forms

$$2.2 \qquad x_i' = \theta x_j, \quad x_j' = \theta^{-1} x_i, \quad x_k' = x_k \qquad (k \neq i, j),$$

or

$$2.3 \qquad x_i' = \phi x_i, \quad x_j' = x_j \qquad (j \neq i),$$

where ϕ is a root of unity. Since $\mathfrak{G}$ is a u.g.g.r., it is generated by transformations of these two types, and since $\mathfrak{G}$ is irreducible, transformations of the former type certainly occur.

The correspondence $T \to \sigma$ where T is the typical transformation 2.1 of $\mathfrak{G}$, determines a homomorphism Θ of $\mathfrak{G}$ into the symmetric group $\mathfrak{S}_n$ of permutations of the suffixes of the variables. Let $\mathfrak{H}$ be the image of $\mathfrak{G}$ under Θ. Then, since $\mathfrak{G}$ is irreducible, $\mathfrak{H}$ is a transitive subgroup of $\mathfrak{S}_n$. It must be generated by the permutations corresponding to reflections in $\mathfrak{G}$. Now 2.2 is mapped onto the transposition (ij) of $\mathfrak{S}_n$, while 2.3 is mapped onto the identity. It follows that $\mathfrak{H}$ is generated by transpositions, and since it is transitive, $\mathfrak{H}$ must coincide with $\mathfrak{S}_n$, and Θ is a homomorphism of $\mathfrak{G}$ *onto* $\mathfrak{S}_n$.

Consider the subgroup $\mathfrak{G}^*$ of $\mathfrak{G}$ generated by the reflections 2.2. Since $\mathfrak{G}^*$, like $\mathfrak{G}$, is mapped homomorphically onto $\mathfrak{S}_n$ by Θ, the subgroups of $\mathfrak{G}^*$ which keep $n - 2$ variables fixed are all conjugate in $\mathfrak{G}^*$. The linear group induced on the

remaining variables, since it leaves the simplex of reference invariant, must be dihedral, of order $2m$ say. It follows that $\mathfrak{G}^*$ is generated by transformations 2.2 where θ is a primitve mth root of unity, and $\mathfrak{G}^*$ is the group, of order $m^{n-1}n!$, of transformations of the form

2.4 $$x_i' = \theta^{\nu_i} x_{\sigma(i)}$$

where

2.5 $$\sum \nu_i \equiv 0 \pmod{m}.$$

If, in addition, $\mathfrak{G}$ contains a transformation of the type 2.3, this must transform the subgroup $\mathfrak{G}^*$ into itself. By considering the effect of 2.3 on the transformations of $\mathfrak{G}^*$ keeping fixed all the variables except x_i and x_j, it follows that ϕ is a power of θ. Thus if 2.3 is a q-fold reflection, so that ϕ is a primitive qth root of unity, q must divide m. Conversely, if $m = pq$, the set of transformations 2.4, where θ is a primitive mth root of unity and

2.6 $$\sum \nu_i \equiv 0 \pmod{p}$$

determine a u.g.g.r. of order $qm^{n-1}n!$ containing, in addition to the 2-fold reflections 2.2, q-fold reflections of type 2.3. This group may be denoted by $G(m, p, n)$. It is defined for all integral m, n ($m > 0$, $n > 1$) and any divisor p of m. But, if $m = 1$, the group leaves fixed the prime $\sum x_i = 0$, and hence is reducible. We therefore suppose that $m > 1$.

$G(m, p, n)$ is the symmetry group of a complex polytope which may be denoted by $\frac{1}{p}\gamma_n^m$ in analogy with the notation of **(24**, p. 378**)**. It has $qm^{n-1} = m^n/p$ vertices:

$$(\theta^{\nu_1}, \theta^{\nu_2}, \ldots, \theta^{\nu_n}), \qquad \sum \nu_i \equiv 0 \pmod{p}.$$

Thus $G(m, 1, n)$ is the symmetry group of γ_n^m, and $G(m, m, n)$ is $[1\,1; n-2]^m$ ($n > 2$) in the notation of **(24)**. $G(m, m, 2)$ is the dihedral group of order $2m$, being the symmetry group of the polygon $\frac{1}{m}\gamma_2^m = \{m\}$, the real regular m-gon **(24**, p. 378**)**.

We have now proved that the only irreducible imprimitive u.g.g.r. are the groups $G(m, p, n)$.

3. Primitive u.g.g.r. in U_n ($n \neq 2$). We now consider the primitive groups. When $n = 1$ it is clear that the only possibility is a cyclic group generated by a single reflection. When $n = 2$ we are led to a number of complications peculiar to this dimension which it will be convenient to discuss separately in the next section. Here we shall consider only the cases in which n exceeds two; these cases are characterised by the fact that the matrix of a homology in S_{n-1} can be normalized *in a unique way* to give a reflection matrix in U_n.

There is one simple general case in which $\mathfrak{G}$ is the symmetric group of order $(n+1)!$, the complete symmetry group of the regular simplex in E_n. The collineation group $\mathfrak{G}'$ in this case is simply isomorphic to $\mathfrak{G}$.

Apart from this, there are only a finite number of irreducible primitive groups generated by homologies. These groups are all known, and many of them have

been extensively studied. We shall list them here, together with the u.g.g.r. to which they give rise. A suffix will be used to denote the order of the group.

When $n = 3$, the groups $\mathfrak{G}'$ generated by homologies are of orders 60, 168, 216 or 360 **(18**, ch. XII**)**:

$\mathfrak{G}'_{60}$ is isomorphic with the alternating group on five symbols, and the corresponding group $\mathfrak{G}$, of order 120, is [3, 5], the symmetry group of the regular icosahedron.

$\mathfrak{G}'_{168}$ is Klein's simple group **(14)** and leads to a u.g.g.r. of order 336 containing 21 2-fold reflections.

$\mathfrak{G}'_{216}$ is the *Hessian group* which leaves invariant the configuration of inflections of a cubic curve. This group is generated by homologies of period 3, belonging to 12 cyclic subgroups, and in addition contains nine homologies of period 2 which generate an imprimitive subgroup. A u.g.g.r. corresponding to this collineation group must contain 3-fold reflections, and may, in addition, contain 2-fold reflections corresponding to the homologies of period 2. There are in fact two such groups; one, of order 648, contains only 3-fold reflections and is the symmetry group of the complex regular polytope $3(24)3(24)3$ (in the notation of Shephard **(23)**) and the other, of order 1296, containing both 3-fold and 2-fold reflections, is the symmetry group of the regular complex polytope $3(24)3(18)2$ (or its reciprocal).

$\mathfrak{G}'_{360}$ is the collineation group first found by Valentiner **(28)** and shown by Wiman **(29)** to be isomorphic with the alternating group of degree six. It leads to a u.g.g.r. of order 2160 generated by 45 2-fold reflections.

The primitive collineation groups in four variables have been given by Blichfeldt **(3)**, Bagnera **(1)** and Mitchell **(20)**. Those which are generated by homologies are of orders 576, 1920, 7200, 11520 and 25920:

$\mathfrak{G}'_{576}$ is the collineation group leaving fixed a pair of associated sets of desmic tetrahedra, and the corresponding u.g.g.r. is Euclidean of order 1152, namely [3, 4, 3], the symmetry group of the regular 24-cell **(9)**.

$\mathfrak{G}'_{7200}$ also gives rise to a Euclidean group, namely [3, 3, 5], of order 14400, the symmetry group of regular 120-cell **(9)**.

$\mathfrak{G}'_{11520}$ is the group leaving invariant Klein's 60_{15} figure, and $\mathfrak{G}'_{1920}$ is a subgroup of this leaving fixed a set of five of the 15 fundamental tetrahedra. To these correspond u.g.g.r. of orders 46080 and 7680 respectively, namely the symmetry groups of the complex polytopes $(\frac{1}{2}\gamma_3{}^4)^{+1}$ and $(\frac{1}{4}\gamma_3{}^4)^{+1}$. The latter is the group $[2\ 1;\ 1]^4$ in Shephard's notation **(24**, p. 373**)**.

$\mathfrak{G}'_{25920}$ is unique among the quaternary groups in that all its homologies are of period three, and it gives rise to a u.g.g.r. of order 155,520, the symmetry group of the complex regular polytope $3(24)3(24)3(24)3$ **(23)**.

The primitive groups generated by homologies in more than four variables were determined by Mitchell **(21)**. There are five of them, which fall into two sets. The first set comprises groups in S_5, S_6 and S_7 respectively of orders 72.6!, 4.9! and 96.10! to which correspond Euclidean groups of orders 72.6!, 8.9! and

192.10! which are the symmetry groups of the polytopes 2_{21}, 3_{21} and 4_{21} respectively **(6)**. The remaining two groups, of orders 36.6! and 18.9! in S_4 and S_5 **(2; 12; 25)** correspond to u.g.g.r. (in U_5 and U_6 respectively) of orders 72.6! and 108.9!, they are the groups $[2\ 1;\ 2]^3$ and $[2\ 1;\ 3]^3$ in the notation of Shephard **(24**, p. 373**)**.

4. Primitive u.g.g.r. in U_2. The enumeration of the primitive u.g.g.r. in U_2 is complicated by the fact that any collineation (other than the identity) of finite period m can be regarded as a homology, and its matrix can be normalised in two ways so as to give a reflection matrix. As far as the collineation groups are concerned the matter is simple enough: the only primitive collineation groups in S_1 are the tetrahedral, octahedral and icosahedral groups. To each of these, however, correspond several u.g.g.r. in U_2.

Let $\mathfrak{G}$ be one of these u.g.g.r. and let $\mathfrak{G}'$ be the corresponding collineation group. Let Z be a generator of the cyclic subgroup $\mathfrak{Z}$ of $\mathfrak{G}$ defined in §1; then $\mathfrak{Z}$ is an invariant subgroup of $\mathfrak{G}$ and $\mathfrak{G}' \cong \mathfrak{G}/\mathfrak{Z}$. Thus if k denotes the period of Z, k operations of $\mathfrak{G}$ correspond to any collineation S' of $\mathfrak{G}'$, and if S is any one of these, the whole set is given by SZ^r $(r = 0, 1, \ldots, k-1)$.

Let S be an m-fold reflection, with matrix $\mathbf{S}$, so that its characteristic roots are $(1, \theta_m)$ where θ_m is a primitive mth root of unity, and let $\mathbf{Z}$ be the matrix corresponding to Z. Then, if θ_k is the primitive kth root of unity such that $\mathbf{Z} = \theta_k \mathbf{I}$, the characteristic roots of $\mathbf{SZ}^r$ are $(\theta_k{}^r, \theta_m\theta_k{}^r)$. If this is a reflection then either $\theta_k{}^r = 1$ (i.e., $r = 0$) or $\theta_m\theta_k{}^r = 1$. This latter case occurs if and only if θ_m is a power of θ_k, i.e., if m is a factor of k. Hence, *if S' is an operation of period m in $\mathfrak{G}'$ which corresponds to an m-fold reflection in $\mathfrak{G}$, there are either one or two such reflections corresponding to S' according as k is not, or is, multiple of m, where k is the order of the subgroup $\mathfrak{Z}$ of $\mathfrak{G}$ represented by scalar matrices.*

If T' is an operation of $\mathfrak{G}'$ conjugate to S', then among the operations of $\mathfrak{G}$ which correspond to T' is one conjugate to S. Hence if one operation of a conjugate set in $\mathfrak{G}'$ has one (or two) corresponding reflections in $\mathfrak{G}$, then the same is true of every operation of the set.

The operation S^{-1} of $\mathfrak{G}$ clearly corresponds to $(S')^{-1}$ in $\mathfrak{G}'$. Suppose now that S' and its inverse are conjugate in $\mathfrak{G}'$, and that $m > 2$ (the theory that follows being trivial if $m = 2$). Then, among the operations $S^{-1}Z^r$ is one conjugate to S in $\mathfrak{G}$. This is a reflection (since it is conjugate to the reflection S) and is distinct from S^{-1}, since the characteristic roots of $\mathbf{S}^{-1}$ are $(1, \theta_m{}^{-1})$ and are different from those of $\mathbf{S}$. Consequently

If S' is an operation of period m $(m > 2)$ in $\mathfrak{G}'$ conjugate to its inverse, which corresponds to an m-fold reflection S, then there are two reflections in $\mathfrak{G}$ which correspond to S' and so (by the last result proved) m is a factor of k.

The method of constructing all u.g.g.r. corresponding to $\mathfrak{G}'$ is then as follows. We start from the abstract definition of $\mathfrak{G}'$ in the well-known form

$$s^2 = t^3 = (st)^p = 1$$

($p = 3, 4, 5$ according as $\mathfrak{G}'$ is tetrahedral, octahedral or icosahedral). We then

take a representation $\mathfrak{G}_1$ of $\mathfrak{G}$ by unitary matrices, for instance, Klein's representation by unimodular matrices (**15**, ch. 2). If $\mathbf{S}_1$ and $\mathbf{T}_1$ are a pair of matrices corresponding to s and t, then $\mathbf{S}_1^2$, $\mathbf{T}_1^3$ and $(\mathbf{S}_1\mathbf{T}_1)^p$ are all scalar matrices. Any u.g.g.r. corresponding to $\mathfrak{G}'$ will then have, as corresponding matrices to s and t, scalar multiples of $\mathbf{S}_1$ and $\mathbf{T}_1$. These multiples are to be chosen in such a way that $\mathfrak{G}$ contains reflection matrices and is generated by them. The sets of possible multipliers are finite in number and can be determined without difficulty. We give explicit results below, but suppress the details of the work involved. It should be noted that here (as elsewhere) we do not regard the group $\mathfrak{G}$ as distinct from its conjugate $\overline{\mathfrak{G}}$ (in which every matrix is replaced by its complex conjugate).

In order to save space, we shall arrange the results in tabular form. For each collineation group $\mathfrak{G}'$ we give explicit forms for Klein's generators $\mathbf{S}_1$, $\mathbf{T}_1$ and $\mathbf{S}_1\mathbf{T}_1$ of the corresponding linear group, together with the values of $\mathbf{S}_1^2$, $\mathbf{T}_1^3$, $(\mathbf{S}_1\mathbf{T}_1)^p$ and the characteristic roots of matrices corresponding to a generator of each cyclic subgroup of $\mathfrak{G}'$. The corresponding u.g.g.r. will then be generated by matrices $\mathbf{S} = \lambda\mathbf{S}_1$, $\mathbf{T} = \mu\mathbf{T}_1$ (where λ, μ are suitably chosen roots of unity) and will have an abstract definition of the form

$$S^2 = Z^{k_1},\quad T^3 = Z^{k_2},\quad (ST)^p = Z^{k_3},\quad ZS = SZ,\quad ZT = TZ,\quad Z^k = 1.$$

The table giving the details of the u.g.g.r. corresponding to each collineation group lists, in its first eight columns, the serial number of the group in the complete list in Table VII, the order of the group, and the values of λ, μ, k_1, k_2, k_3, k. Then follow columns giving the number r_q of q-fold reflections in $\mathfrak{G}$, for various values of q. Typical reflections may easily be identified by considering the characteristic roots of the products of $\mathbf{S}$, $\mathbf{T}$, $(\mathbf{ST})^j$ with powers of $\mathbf{Z}$. The multipliers λ, μ have been chosen so that Z is always represented by the matrix $\exp(2\pi i/k)\,\mathbf{I}$.

The groups obtained are of two kinds according as they can be generated by two reflections, or require three. When $\mathfrak{G}$ can be generated by two reflections it is the symmetry group of a complex regular polygon, whose symbol (in the notation of (**23**)) is given in the table, together with explicit expressions for a pair of generating reflections. These are not necessarily the simplest pair, but have been chosen in such a way as to verify a property of these groups given later (see 5.4).

Groups derived from the tetrahedral group.

$$\mathbf{S}_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix},\quad \mathbf{T}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon^3 \\ \epsilon & \epsilon^7 \end{pmatrix},\quad \mathbf{S}_1\mathbf{T}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon^3 & \epsilon^5 \\ \epsilon^7 & \epsilon^5 \end{pmatrix} \qquad (\epsilon = \exp(2\pi i/8)).$$

$$\mathbf{S}_1^2 = \mathbf{T}_1^3 = -\mathbf{I},\quad (\mathbf{S}_1\mathbf{T}_1)^3 = \mathbf{I}.$$

The characteristic roots of $\mathbf{S}_1$, $\mathbf{T}_1$, $\mathbf{S}_1\mathbf{T}_1$ are $(i, -i)$, $(-\omega, -\omega^2)$, (ω, ω^2) respectively.

The four corresponding groups $\mathfrak{G}$ are shown in Table I.

TABLE I

No.	Order	λ	μ	k_1	k_2	k_3	k	r_2	r_3	Polygon	Pair of generating reflections
4	24	-1	$-\omega$	1	2	2	2		8	$3(24)3$	ST, T
5	72	$-\omega$	$-\omega$	1	6	6	6		16	$3(72)3$	T, $(ST)^{-1}$
6	48	i	$-\omega$	4	4	1	4	6	8	$3(48)2$	SZ^2, T
7	144	$i\omega$	$-\omega$	8	12	3	12	6	16		

Groups derived from the octahedral group.

$$\mathbf{S}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, \quad \mathbf{T}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon \\ \epsilon^3 & \epsilon^7 \end{pmatrix}, \quad \mathbf{S}_1\mathbf{T}_1 = \begin{pmatrix} \epsilon^3 & \\ & \epsilon^5 \end{pmatrix} \quad (\epsilon = \exp(2\pi i/8)).$$

$$\mathbf{S}_1^2 = \mathbf{T}_1^3 = (\mathbf{S}_1\mathbf{T}_1)^4 = -\mathbf{I}.$$

The characteristic roots of $\mathbf{S}_1$, $\mathbf{T}_1$, $\mathbf{S}_1\mathbf{T}_1$, $(\mathbf{S}_1\mathbf{T}_1)^2$ are $(i, -i)$, $(-\omega, -\omega^2)$, (ϵ^3, ϵ^5), $(i, -i)$ respectively.

The eight corresponding groups $\mathfrak{G}$ are shown in Table II.

TABLE II

No.	Order	λ	μ	k_1	k_2	k_3	k	r_2	r_3	r_4	Polygon	Pair of generating reflections
8	96	ϵ^3	1	1	2	4	4	6		12	$4(96)4$	TS, ST
9	192	i	ϵ	8	7	8	8	18		12	$4(192)2$	S, ST
10	288	$\epsilon^7\omega^2$	$-\omega$	7	12	12	12	6	16	12	$4(288)3$	ST, $(TZ^4)^{-1}$
11	576	i	$\epsilon\omega$	24	21	8	24	18	16	12		
12	48	i	1	2	1	1	2	12				
13	96	i	i	4	1	2	4	18				
14	144	i	$-\omega$	6	6	5	6	12	16		$3(144)2$	SZ^2, T
15	288	i	$i\omega$	12	3	10	12	18	16			

It should be noticed that the two groups (8) and (13), of the same order 96 are not isomorphic, for the squares of the operations of period eight (e.g. S) in the former group belong to the central, while the squares of the operations of period eight (e.g. ST) in the latter do not. Similarly the groups (10) and (15), each of order 288, are distinct abstract groups (being isomorphic with the direct products of the two former groups with a cyclic group of order three).

Groups derived from the icosahedral group.

$$\mathbf{S}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^4 - \eta & \eta^2 - \eta^3 \\ \eta^2 - \eta^3 & \eta - \eta^4 \end{pmatrix}, \quad \mathbf{T}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^2 - \eta^4 & \eta^4 - 1 \\ 1 - \eta & \eta^3 - \eta \end{pmatrix}, \quad \mathbf{S}_1\mathbf{T}_1 = \begin{pmatrix} -\eta^3 & \\ & -\eta^2 \end{pmatrix}$$

$$(\eta = \exp(2\pi i/5)).$$

$$\mathbf{S}_1^2 = -\mathbf{I}, \ \mathbf{T}_1^3 = \mathbf{I}, \ (\mathbf{S}_1\mathbf{T}_1)^5 = -\mathbf{I}.$$

The characteristic roots of $\mathbf{S}_1$, $\mathbf{T}_1$, $\mathbf{S}_1\mathbf{T}_1$ are $(i, -i)$, (ω, ω^2), $(-\eta^3, -\eta^2)$ respectively.

The seven corresponding groups $\mathfrak{G}$ are shown in Table III.

TABLE III

No.	Order	λ	μ	k_1	k_2	k_3	k	r_2	r_3	r_5	Polygon	Pair of generating reflections
16	600	$-\eta^3$	1	7	10	10	10			48	*5(600)5*	$(ST)^{-1}, (TS)^{-1}$
17	1200	i	$i\eta^3$	20	11	20	20	30		48	*5(1200)2*	$(ST)^{-1}, (SZ^{10})^{-1}$
18	1800	$-\omega\eta^3$	ω^2	11	30	30	30		40	48	*5(1800)3*	$TZ^{20}, (ST)^{-1}$
19	3600	$i\omega$	$i\eta^3$	40	33	40	60	30	40	48		
20	360	1	ω^2	3	6	5	6		40		*3(360)3*	$ST^{-1}S^{-1}, T^{-1}$
21	720	i	ω^2	12	12	1	12	30	40		*3(720)2*	T^{-1}, S^{-1}
22	240	i	1	4	4	3	4	30				

This completes our determination of all the irreducible u.g.g.r. A complete list appears in Table VII on page 301.

II. SOME PROPERTIES OF FINITE UNITARY GROUPS GENERATED BY REFLECTIONS

5. Statement of results to be proved. The next five sections (§§6-10) of this paper are devoted to establishing the following propositions:

5.1 *Let $\mathfrak{G}$ be a finite group, of order g, of unitary transformations on n variables. Then the following properties are equivalent:*

(a) *$\mathfrak{G}$ is a u.g.g.r.*

(b) *$\mathfrak{G}$ possesses a set of n algebraically independent invariant forms $I_1, I_2, \ldots, I_n$ of degrees $m_1 + 1, m_2 + 1, \ldots, m_n + 1$, such that*

$$\prod_{i=1}^{n} (m_i + 1) = g.$$

(c) *There exists a set of n algebraically independent invariant forms of $\mathfrak{G}$ such that every invariant form of $\mathfrak{G}$ is expressible as a polynomial in the forms of the set.* (Such a set of forms will be called a *basic* set.)

The integers $m_1, m_2, \ldots, m_n$, which we shall call the *exponents* of the group, are determined uniquely. The degrees of the forms of any basic set satisfy the

condition (b), and any set of n algebraically independent forms which satisfies (b) is a basic set. The number of reflections in $\mathfrak{G}$ is Σm_i.

It will be convenient in what follows to suppose that

$$m_1 \leqslant m_2 \leqslant \ldots \leqslant m_n.$$

5.2 *If $\mathfrak{G}$ is a u.g.g.r. in U_n, then the Jacobian of the n forms of a basic set with respect to the variables $x_1, x_2, \ldots, x_n$ factorises into the product of the reflecting primes, the prime corresponding to a p-fold reflection counting with multiplicity $p - 1$.*

5.3 *If $\mathfrak{G}$ is a u.g.g.r. in U_n, the number of operations which leave fixed all the points of some linear space of dimension $n - k$ is the coefficient of t^k in the product*

$$\prod_{i=1}^{n} (1 + m_i t)$$

where $m_1, m_2, \ldots, m_n$ are the exponents of the group.

5.4 *If $\mathfrak{G}$ is an irreducible u.g.g.r. in U_n which is generated by n reflections, then these reflections can be chosen and ordered in such a way that their product has period $h = m_n + 1$, and so that the characteristic roots of this product are*

$$\exp(2\pi i m_r/h) \qquad (r = 1, 2, \ldots, n),$$

where the m_i are the exponents of the group.

Before proceeding to the proofs of these propositions it is convenient to discuss them in more detail.

Of these theorems, 5.1 is perhaps of the greatest theoretical interest since it gives a characterisation of the u.g.g.r. in terms of their invariants. In the real case, when the transformations are orthogonal, Coxeter has shown **(10)** that (a) implies (b), and that any invariant form of $\mathfrak{G}$ can be expressed rationally in terms of n forms satisfying condition (b). But it does not appear from Coxeter's argument that this expression is necessarily integral. Coxeter also obtains Σm_i for the number of reflections in the real case. Our proof of 5.1, which occupies §§6–8, runs along the following lines. It is clear that condition (a) implies (b) for a reducible u.g.g.r. if it does so for each irreducible component. Since, from Coxeter's results, it holds for the Euclidean groups, it is sufficient to verify it for each of the irreducible unitary g. g. r. which are not equivalent to orthogonal groups. This verification occupies §6, and is based either on explicit construction of a set of invariant forms, or upon known results for the associated collineation groups. Next, in §7, we prove that (b) implies (c) by showing that any set of n invariant forms which satisfy (b) is necessarily basic. Since the degrees of the invariant forms of a basic set are evidently unique, this establishes the invariance of the exponents m_i. Finally, in §8, we prove that (c) implies (a), and show in the course of the proof that the number of reflections of $\mathfrak{G}$ is Σm_i.

It is of interest to note, in this connection, that Frame has shown **(10a)** how these basic invariant forms may be used to compute the character table of the group.

Coxeter gives **(10)**, for the real case, a deduction of 5.2 from the first part of the proof of 5.1 by a very simple argument due to Racah. This argument extends

at once to the unitary groups; the additional complication due to multiplicities greater than unity if the group contains p-fold reflections with $p > 2$ is not serious. We omit details of this proof.

Theorem 5.3 implies that the number of reflections in $\mathfrak{G}$ is Σm_i and that the order of $\mathfrak{G}$ is $\Pi(m_i + 1)$, both of which results are implicit in 5.1. We have been unable to extend the argument to deduce the remaining properties expressed by 5.3 from 5.1. Since 5.3 holds for reducible groups if it holds for each irreducible component, it is enough to verify the result for each of the irreducible groups listed in Table VII. This is done in §9 and Table VIII.

The result expressed by 5.4 is in a somewhat different category. In the first place it only applies to a restricted class of u.g.g.r., since both the condition of irreducibility and the condition of being generated by n reflections (and not $n + 1$) are relevant. In the real case, a more precise result has been given by Coxeter **(6; 10)**, who shows that the various products formed from n generating reflections arranged in different orders are all conjugate in $\mathfrak{G}$, and therefore have the same characteristic roots. His argument **(6**, p. 602**)** depends upon the fact that the graph **(9**, p. 84**)** of any real reflection group is a *tree*, i.e., contains no *circuits*. When the group is unitary the graph must contain **(24**, p. 368**)** either a circuit or a numbered node (i.e., a p-fold reflection with $p > 2$) and so Coxeter's argument is no longer valid. In general, therefore, the various products of generators of a unitary g. g. r. in different orders are not necessarily conjugate, and it is necessary to select the appropriate generators and their order. Consequently, to establish 5.4 we write down a set of generating reflections which have the required property. This is done in §10.

For a real group the exponents occur in pairs such that

$$m_r + m_{n-r} = h \qquad (r = 1, 2, \ldots, [\tfrac{1}{2}n]).$$

This is not true for unitary groups in general, nor is the corollary $\Sigma m_i = \frac{1}{2}nh$ **(10**, p. 772**)**.

6. The invariants of the irreducible u.g.g.r. In this section we consider in turn the irreducible u.g.g.r. which are not expressible as groups of orthogonal transformations in Euclidean space, and show that in each case we can find an algebraically independent set of invariants, the product of whose degrees is equal to the order of the group $\mathfrak{G}$. The groups are numbered as in Table VII.

(2) Reference to the explicit form of the operations of $G(m, p, n)$ in §2 shows that a suitable set of invariants consists of the elementary symmetric functions of $x_1^m, x_2^m, \ldots, x_n^m$ of degrees $1, 2, \ldots, n-1$, together with $(x_1x_2 \ldots x_n)^e$.

(3) The cyclic group of order m in U_1 is generated by a single m-fold reflection, and so there is clearly a single invariant, which may be taken to be x_1^m.

Consider next the groups in U_2. The invariant theory of the binary linear groups is given in detail by Klein **(15)**. For each of the three collineation groups (tetrahedral, octahedral, icosahedral) there are three invariant forms which may be denoted by f, h, t; h is the Hessian of f, and t the Jacobian of f and h. Any invariant of the collineation group is expressible integrally and rationally in terms of f, h and t, and there is a syzygy expressing t^2 as a polynomial in f and

h. These forms are relative invariants of any linear group which has this collineation group associated with it. The determination of the absolute invariants is easily accomplished by examining the effect on the various forms of a pair of generating operations of $\mathfrak{G}$.

For the tetrahedral group f, h, t are of respective degrees 4, 4, 6, and t^2 is a linear combination of f^3 and h^3. Under the generating transformations S_1 and T_1 of the binary tetrahedral group given in §4, the forms f, h, t are absolutely unaltered by S_1, and receive factors ω^2, ω, 1 respectively under the operation T_1. (Precisely this means that if T_1 is represented as a transformation $x' = Tx$, then $f(x'_1, x'_2) \equiv \omega^2 f(x_1, x_2)$ and so on). We now examine each of the groups of Table I which have the tetrahedral group as the corresponding collineation group and note the effect on f, h, t of the generators S, T of the u.g.g.r. Hence we may pick out the invariant forms of the group. The results are listed in Table IV.

TABLE IV

Group	Order	Sf	Sh	St	Tf	Th	Tt	Invariants	Degrees
4	24	1	1	1	1	ω^2	1	f, t	4, 6
5	72	ω	ω	1	1	ω^2	1	f^3, t	12, 6
6	48	1	1	-1	1	ω^2	1	f, t^2	4, 12
7	144	ω	ω	-1	1	ω^2	1	f^3, t^2	12, 12

The columns headed Sf, Sh, St, Tf, Th, Tt give the factors by which the forms f, h, t are multiplied under operation by S or T.

For the octahedral group, f, h, t have respective orders 6, 8, 12, and under S_1 and T_1 are multiplied by $(-1, 1, -1)$ and $(1, 1, 1)$ respectively. Hence the results for the groups associated with the octahedral group are as listed in Table V.

TABLE V

Group	Order	Sf	Sh	St	Tf	Th	Tt	Invariants	Degrees
8	96	$-i$	1	1	1	1	1	h, t	8, 12
9	192	1	1	-1	$-i$	1	-1	h, t^2	8, 24
10	288	$-i$	ω	1	1	ω^2	1	h^3, t	24, 12
11	576	1	1	-1	$-i$	ω^2	-1	h^3, t^2	24, 24
12	48	1	1	-1	1	1	1	f, h	6, 8
13	96	1	1	-1	-1	1	1	f^2, h	12, 8
14	144	1	1	-1	1	ω^2	1	f, t^2	6, 24
15	288	1	1	-1	-1	ω^2	1	f^2, t^2	12, 24

For the icosahedral group, f, h, t have respective orders 12, 20, 30 and are absolute invariants under S_1 and T_1. Hence the results for the remaining groups in U_2 are as listed in Table VI, in which $\eta = \exp(2\pi i/5)$.

TABLE VI

Group	Order	Sf	Sh	St	Tf	Th	Tt	Invariants	Degrees
16	600	η	1	1	1	1	1	h, t	20, 30
17	1200	1	1	-1	η	1	-1	h, t^2	20, 60
18	1800	η	ω^2	1	1	ω	1	h^3, t	60, 30
19	3600	1	ω^2	-1	η	1	-1	h^3, t^2	60, 60
20	360	1	1	1	1	ω	1	f, t	12, 30
21	720	1	1	-1	1	ω	1	f, t^2	12, 60
22	240	1	1	-1	1	1	1	f, h	12, 20

The results for the u.g.g.r. in U_n with $n > 2$ are obtained (with one exception, the group of order 108.9! in U_6) as follows. Each such group $\mathfrak{G}$ corresponds to a collineation group $\mathfrak{G}'$ whose invariants have been determined explicitly (we shall give precise references below). To determine the invariants of $\mathfrak{G}$ it is therefore sufficient to examine the behaviour of these invariants of $\mathfrak{G}'$ under the generating reflections of $\mathfrak{G}$. These operations correspond to homologies in $\mathfrak{G}'$ which form a single conjugate set of cyclic subgroups (except in the case of the Hessian group, where there are two sets of homologies). Since conjugate operations of $\mathfrak{G}$ have the same effect on any relative invariant, and since the conjugate cyclic subgroups corresponding to the homologies in $\mathfrak{G}'$ determine conjugate cyclic subgroups corresponding to the reflections in $\mathfrak{G}$, it is sufficient to consider the behaviour of the forms for a single reflection in $\mathfrak{G}$ [or for a single reflection of each of the two types in the case of the symmetry group of $3(24)3(18)2$]. The classical expressions for these groups in each case possess a homology whose equations are given by a matrix of very simple (diagonal or monomial) form, and so the necessary verification is almost immediate. It seems unnecessary to give further details.

(24) The invariants of the collineation group of order 168 in S_2 consist **(14)** of a quartic, its Hessian (a sextic), a covariant of degree 14, and the Jacobian of these, which breaks up into the 21 axes of the homologies in the group. For the corresponding u.g.g.r. of order 336 in U_3, the forms of degrees 4, 6, 14 are absolute invariants, while their Jacobian changes sign for half the operations of the group **(5; 18)**.

(25) and (26) The Hessian group of order 216 in S_2 is the collineation group that leaves invariant the nine inflections of a pencil of cubic curves in the plane. The invariant forms of this collineation group comprise a sextic $I_{(6)}$, a form $I_{(9)}$ of degree 9 representing the harmonic polars of the inflections, and two forms

$I_{(12)}$, $I'_{(12)}$ representing respectively the four degenerate cubics and the four equianharmonic cubics in the pencil. A syzygy connects $I_{(12)}^3$ with $I_{(6)}$, $I_{(9)}$ and $I'_{(12)}$ **(16; 18,** p. 253**)**. For the two corresponding u.g.g.r. in U_3 of order 648 and 1296 respectively, the invariant forms may be taken to be $I_{(6)}$, $I_{(9)}$, $I'_{(12)}$ and $I_{(6)}$, $I'_{(12)}$, $I_{(18)}$ respectively, where

$$I_{(18)} = 432\, I_{(9)}^2 - I_{(6)}^3 + 3I_{(6)}I'_{(12)}.$$

The Jacobian of $I_{(6)}$, $I_{(9)}$, $I'_{(12)}$ is a multiple of the square of $I_{(12)}$; that of $I_{(6)}$, $I'_{(12)}$, $I_{(18)}$ is a multiple of the product $I_{(9)}I_{(12)}^2$.

(27) The collineation group of order 360 in S_2 has invariant forms of degrees 6, 12, 30 with a Jacobian of degree 45 **(29; 18,** p. 254**)**, related in the same way as the forms corresponding to the group of order 168. These forms, of degrees 6, 12, 30 are invariant for the corresponding u.g.g.r. of order 2160 in U_3.

(29) and (31) The invariant forms for the collineation group of order 11520 in S_3 are of degrees 8, 12, 20, 24 and 60, the latter form being the Jacobian of the first four and having reducible square. The form of degree 24 can be taken to be the product of six quartic forms, and the collineation group of order 1920 in S_3 is the group keeping one of these forms fixed; its fundamental system consists of invariants of degrees 4, 8, 12, 20 together with their Jacobian (of degree 40). The forms of degrees 8, 12, 20, 24 are invariants for the group (31), and those of degrees 4, 8, 12, 20 for (29) **(16)**.

(32) For the collineation group of order 25920 in S_3, Maschke **(17)** obtains invariants of degrees 12, 18, 24 and 30, the Jacobian of these being the square of a form of degree 40 representing the primes of the 40 homologies of period three contained in the group. These forms of degrees 12, 18, 24, 30 are invariant for the corresponding u.g.g.r. (32).

(33) For the collineation group of order 25920 in S_4 Burkhardt **(4)** obtains invariants of degrees 4, 6, 10, 12, 18. These forms are invariants for the corresponding u.g.g.r. of order 51840. Their Jacobian, of degree 45, consists of the invariant primes of the 45 homologies in $\mathfrak{G}'$.

(34) Finally, the group $[2\ 1;\ 3]^3$ of order 108.9! in U_6 possesses a system of invariants of degrees 6, 12, 18, 24, 30, 42 whose Jacobian is of degree 126. The existence of these forms was indicated by Todd **(26)** and the slight reservation expressed there about their possible interdependence can be settled by a calculation like that made by Coxeter **(10,** p. 777**)** showing that for a certain special set of values of the variables the Jacobian of these forms does not vanish.

The remaining groups, which are all Euclidean, have been discussed by Coxeter **(10)**. In Table VII we summarise the results, by listing, in the seventh column of the table, the degrees of the invariant forms for all the irreducible u.g.g.r. It is easily verified that their product is equal to the order of the group and so we conclude:

Any irreducible u.g.g.r. in U_n possesses a set of n algebraically independent invariant forms, the product of whose degrees is equal to the order of the group.

The extension of this result to reducible groups is almost immediate.

7. Basic sets of invariants. In this section we prove that the condition 5.1 (b) implies 5.1 (c). More precisely, we prove the following:

Let $\mathfrak{G}$ be a finite group of linear substitutions on n variables, and suppose that $\mathfrak{G}$ possesses a set of n algebraically invariant forms $I_1, I_2, \ldots, I_n$ of degrees $m_1 + 1$, $m_2 + 1, \ldots, m_n + 1$ such that $\Pi(m_i + 1) = g$, the order of $\mathfrak{G}$. Then any invariant form of $\mathfrak{G}$ is expressible rationally and integrally in terms of $I_1, I_2, \ldots, I_n$.

In the first place we show that any invariant form of $\mathfrak{G}$ may be expressed *rationally* in terms of $I_1, I_2, \ldots, I_n$. This may be done (exactly as in **(10)**) by observing that the solutions of the equations

$$I_i(x_1, x_2, \ldots, x_n) = c_i \qquad (i = 1, 2, \ldots, n)$$

where the c_i are constants, form a *single* set of conjugate points under $\mathfrak{G}$ (if the constants are sufficiently general). Consequently any invariant of $\mathfrak{G}$, necessarily an algebraic function of $I_1, I_2, \ldots, I_n$ is a single-valued, and hence a rational function of these forms.

Let $(\xi_1, \xi_2, \ldots, \xi_n)$ be a point of the space of the n variables $(x_1, x_2, \ldots, x_n)$ whose transforms under $\mathfrak{G}$ are all distinct and which therefore does not make the Jacobian of $I_1, I_2, \ldots, I_n$ vanish. Let $\beta_i = I_i(\xi_1, \xi_2, \ldots, \xi_n)$. Then the system of equations

7.1 $$I_i(x_1, x_2, \ldots, x_n) = \beta_i$$

has a system of $g = \Pi(m_i + 1)$ distinct solutions, and these solutions are isolated, since the Jacobian of $I_1, I_2, \ldots, I_n$ does not vanish for any of them. It follows from the general theory of elimination that these exhaust all the solutions of 7.1, and in particular there are no 'infinite' solutions, (i.e., solutions of the homogeneous equations

$$I_i(x_1, x_2, \ldots, x_n) = \beta_i x_0^{m_i+1}$$

with $x_0 = 0$). Consequently *for any finite set of constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ the equations*

7.2 $$I_i(x_1, x_2, \ldots, x_n) = \alpha_i \qquad (i = 1, 2, \ldots, n)$$

have at least one finite solution (no 'infinite solutions' being possible since these would also be solutions of 7.1).

Now suppose that $J(x_1, x_2, \ldots, x_n)$ is an invariant of $\mathfrak{G}$ that is expressible *rationally*, but not *integrally*, in terms of $I_1, I_2, \ldots, I_n$. We shall show that this assumption leads to a contradiction.

Let $\phi(y_1, y_2, \ldots, y_n)$ and $\psi(y_1, y_2, \ldots, y_n)$ be polynomials with no common factor, such that

7.3 $$\phi(I_1, I_2, \ldots, I_n) \equiv \psi(I_1, I_2, \ldots, I_n)\ J(x_1, x_2, \ldots, x_n)$$

identically in $x_1, x_2, \ldots, x_n$. By our assumption $\psi(y_1, y_2, \ldots, y_n)$ is of degree greater than zero, that is, is not a constant. Let $\psi_1(y_1, y_2, \ldots, y_n)$ be an irreducible factor of $\psi(y_1, y_2, \ldots, y_n)$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any set of constants such that

$$\psi_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0.$$

Let $x'_1, x'_2, \ldots, x'_n$ be a solution (which we have shown exists) of 7.2. Then, for $x_i = x'_i$, the right hand side of 7.3 is zero, since $\psi_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$. Hence

$$\phi(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$$

since $x'_1, x'_2, \ldots, x'_n$ satisfy 7.2. Thus $\phi(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$ whenever $\psi_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$, and so $\psi_1(y_1, y_2, \ldots, y_n)$ is a factor of $\phi(y_1, y_2, \ldots, y_n)$. This is a contradiction. Thus 7.3 can hold only if $\psi(y_1, y_2, \ldots, y_n)$ is a constant, i.e., if $J(x_1, x_2, \ldots, x_n)$ is a polynomial in $I_1, I_2, \ldots, I_n$.

8. Characterisation of the u.g.g.r. In this section we complete the proof of Theorem 5.1 by proving that (c) implies (a). That is, we prove:

Any finite group of unitary transformations on n variables which possesses a basic set of invariant forms $I_1, I_2, \ldots, I_n$ is a u.g.g.r.

Let $\mathfrak{G}$ be such a group, g its order, and $m_i + 1$ the degree of $I_i (i = 1, 2, \ldots, n)$. The proof depends on a theorem of Molien **(22; 5,** p. 300**)** which states that, in any group of linear transformations, if a_r denote the number of linearly independent invariant forms of $\mathfrak{G}$ of order r, then

$$\sum_0^\infty a_r \lambda^r \equiv \frac{1}{g} \sum_S \frac{1}{(1 - \omega_1^S \lambda) \ldots (1 - \omega_n^S \lambda)} \equiv \frac{1}{g} \sum_S \frac{1}{|\mathbf{I} - \mathbf{S}\lambda|}$$

where $\omega_1^S, \omega_2^S, \ldots, \omega_n^S$ denote the characteristic roots of the operation S of $\mathfrak{G}$, and the sums on the right extend over all the operations of the group.

Now if the invariants $I_1, I_2, \ldots, I_n$ form a basic set (so that they are algebraically independent, and every invariant can be expressed as a polynomial in them) then the linearly independent forms of degree r may be taken to be just the power products of $I_1, I_2, \ldots, I_n$ of degree r in the variables. Thus the function

$$\prod_{i=1}^n (1 - \lambda^{m_i+1})^{-1}$$

is also a generating function for the number of independent invariant forms of a given degree and so,

8.1 $$g \prod_{i=1}^n \frac{1}{(1 - \lambda^{m_i+1})} \equiv \sum_S \frac{1}{(1 - \omega_1^S \lambda) \ldots (1 - \omega_n^S \lambda)}.$$

On the right of 8.1 there is only one term with denominator $(1 - \lambda)^n$, corresponding to the identical element of $\mathfrak{G}$. Hence, on multiplying both sides of 8.1 by $(1 - \lambda)^n$ and then putting $\lambda = 1$ we obtain

8.2 $$g = \prod_{i=1}^n (m_i + 1).$$

This proves, incidentally, that the product of the degrees of a basic set of invariant forms is necessarily equal to the order of the group.

We now subtract $(1 - \lambda)^{-n}$ from each side of 8.1, multiply by $(1 - \lambda)^{n-1}$ and again put $\lambda = 1$. After a brief calculation we find that the left hand side

reduces to $\frac{1}{2}\Sigma m_i$. The only terms on the right which can contribute non-zero terms to the sum are those whose denominators involve the factor $(1-\lambda)^{n-1}$. These terms correspond to operations S with $n-1$ characteristic roots equal to unity, i.e., to *unitary reflections*. Suppose that π is the fixed prime corresponding to such a reflection, then the aggregate of all reflections for which π is the fixed prime are the elements (other than the identity) of a cyclic group of some order p say, generated by a p-fold reflection S with $n-1$ characteristic roots unity and the other one a primitive pth root of unity. The sum of the corresponding terms in the expression under consideration is thus

$$(1-\theta)^{-1} + (1-\theta^2)^{-1} + \ldots + (1-\theta^{p-1})^{-1} = \tfrac{1}{2}(p-1)$$

where $\theta = \exp(2\pi i/p)$. By summing over all the reflecting primes we obtain $\frac{1}{2}\Sigma m_i = \frac{1}{2}\Sigma(p-1)$. Hence *the number of reflections in* $\mathfrak{G}$ *is equal to* Σm_i.

Let $\mathfrak{H}$ be the subgroup of $\mathfrak{G}$ generated by these reflections. Then $\mathfrak{H}$ is a u.g.g.r., and consequently, by the results of §§6, 7, possesses a basic set of n invariant forms $J_1, J_2, \ldots, J_n$ of degrees $\mu_1+1, \mu_2+1, \ldots, \mu_n+1$, such that $\Pi(\mu_i+1) = h$, the order of $\mathfrak{H}$. Since $\mathfrak{H}$ possesses this basic set of invariant forms, the number of reflections in $\mathfrak{H}$ is $\Sigma\mu_i$ and so

$$\sum \mu_i = \sum m_i. \tag{8.3}$$

Now each of the forms $I_1, I_2, \ldots, I_n$, being invariant for $\mathfrak{G}$, is invariant for the subgroup $\mathfrak{H}$, and is therefore expressible as a polynomial in $J_1, J_2, \ldots, J_n$ (since these last forms constitute a basic set for $\mathfrak{H}$). Hence we may write

$$I_i = \phi_i(J_1, J_2, \ldots, J_n) \qquad (i = 1, 2, \ldots, n)$$

where the ϕ_i are polynomials. Since each of the sets $I_1, I_2, \ldots, I_n$ and $J_1, J_2, \ldots, J_n$ are algebraically independent forms in $x_1, x_2, \ldots, x_n$, the Jacobian of $I_1, I_2, \ldots, I_n$ with respect to $J_1, J_2, \ldots, J_n$ is not identically zero. Consequently there is a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $(1, 2, \ldots, n)$ such that the term

$$\frac{\partial I_{\sigma_1}}{\partial J_1}\,\frac{\partial I_{\sigma_2}}{\partial J_2}\cdots\frac{\partial I_{\sigma_n}}{\partial J_n}$$

is not zero. This implies

$$m_{\sigma_i} + 1 \geqslant \mu_i + 1 \qquad (i = 1, 2, \ldots, n)$$

and so, by 8.3, $m_{\sigma_i} = \mu_i$. Therefore the order $\Pi(\mu_i+1)$ of $\mathfrak{H}$ is equal to the order $\Pi(m_i+1)$ of $\mathfrak{G}$, so that $\mathfrak{H}$ must coincide with $\mathfrak{G}$. The group $\mathfrak{G}$ is therefore generated by the reflections it contains.

This completes the proof of Theorem 5.1.

9. Types of operation in u.g.g.r. In this section we prove Theorem 5.3. This theorem may be restated in the form

$$g_r = s_r(m_1, m_2, \ldots, m_n)$$

where g_r is the number of operations of $\mathfrak{G}$ that leave a space of $n-r$ dimensions

invariant, and s_r is the value of the elementary symmetric function of degree r in the exponents of $\mathfrak{G}$.

We are unable to give an explanation of this property in general terms, except for $r = 1$, which has already been discussed in §8, and for the trivial case $r = 0$ corresponding to the identity of $\mathfrak{G}$. Consequently it is necessary to verify the theorem for each group in Table VII.

This verification is divided, for convenience, into three parts:

(a) Groups $[3^{n-1}]$ and $G(m, p, n)$ in U_n.

(b) The u.g.g.r. in U_2.

(c) The u.g.g.r. in U_n $(n > 2)$ except those in (a).

We proceed to examine each of these in turn.

(a) *The groups* $[3^{n-1}]$ *and* $G(m, p, n)$ *in* U_n.

The group $G(m, p, n)$ consists (cf. §2) of the transformations

$$x'_i = \theta^{\nu_i} x_{\sigma_i} \tag{9.1}$$

where $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is an arbitrary permutation σ of $(1, 2, \ldots, n)$, θ is a primitive mth root of unity, and

$$\nu_1 + \nu_2 + \ldots + \nu_n \equiv 0 \pmod{p}. \tag{9.2}$$

As we have seen, the exponents m_i for this group take the values

$$rm - 1 \quad (r = 1, 2, \ldots, n-1), \qquad nq - 1, \tag{9.3}$$

where $q = m/p$. When $m = p = 1$, the group reduces to the symmetry group of the regular simplex α_{n-1} lying in the prime $\Sigma x_i = 0$. Considered as a group in U_n, Σx_i is a linear invariant and the corresponding exponent m_i is zero. (More pedantically, $G(1, 1, n)$ is the direct product of $[3^{n-1}]$ with a cyclic group of order one in U_1, consisting of the identical operation!) Our treatment will therefore apply, not only to $G(m, p, n)$ but also to the group $[3^{n-1}]$.

Let $g_r(m, p, n)$ denote the number of operations of $G(m, p, n)$ which leave fixed every point of some space of dimension $n - r$ $(r = 0, 1, \ldots, n)$, so that $g_0(m, p, n) = 1$, and let

$$G(m, p, n; t) \equiv \sum_{r=0}^{n} g_r(m, p, n)\, t^r.$$

Any operation leaving fixed every point of a space of dimension $n - r$ has all but r of its characteristic roots equal to unity, and conversely. By forming the characteristic equation of the matrix of transformation 9.1 we see that this will happen if and only if there are exactly $n - r$ cycles in the permutation σ with the property that the sum of the corresponding exponents ν_i is congruent to zero (mod m).

Let us consider a particular permutation σ with $n - r + s$ cycles, $n - r$ of which are designated in advance as the ones for which the exponent sum is congruent to zero (mod m). Let $\rho_1, \rho_2, \ldots, \rho_s$ be the residues of the exponent sums in the remaining s cycles. Then evidently, all but one of the exponents in each cycle can be chosen at will; the last exponent is then completely determined. Thus, for given $\rho_1, \rho_2, \ldots, \rho_s$ there are m^{r-s} ways of choosing the exponents ν_i.

The residues $\rho_1, \rho_2, \ldots, \rho_s$ are subject to the conditions

$$\rho_1 \not\equiv 0, \rho_2 \not\equiv 0, \ldots, \rho_s \not\equiv 0 \pmod{m},$$
$$\rho_1 + \rho_2 + \ldots + \rho_s \equiv 0 \pmod{p},$$

the latter arising from 9.2. Let $\phi(s)$ be the number of solutions of this set of congruences. If $\rho_1, \rho_2, \ldots, \rho_{s-1}$ are given arbitrary values not congruent to zero (mod m), then ρ_s is determined modulo p. If $\rho_1 + \rho_2 + \ldots + \rho_{s-1} \not\equiv 0 \pmod{p}$ then $\rho_s \not\equiv 0 \pmod{p}$ and so there are $m/p = q$ possible values for ρ_s, while if $\rho_1 + \rho_2 + \ldots + \rho_{s-1} \equiv 0 \pmod{p}$ then there are only $q - 1$ possible values (since $\rho_s \equiv 0 \pmod{m}$ is excluded). Consequently

$$\phi(s) = q(m-1)^{s-1} - \phi(s-1)$$

and, since $\phi(1) = q - 1$, we find that

$$\begin{aligned}\phi(s) &= q[(m-1)^{s-1} - (m-1)^{s-2} + \ldots + (-1)^s (m-1)] - (-1)^s [q-1],\\ &= q[(m-1)^{s-1} - (m-1)^{s-2} + \ldots + (-1)^{s-1}] + (-1)^s,\\ &= q[(m-1)^s - (-1)^s]/m + (-1)^s,\\ &= \frac{(m-1)^s + (-1)^s (p-1)}{p}.\end{aligned}$$

It follows that

$$g_r(m, p, n) = \sum_{s=0}^{r} \binom{n-r+s}{s} m^{r-s} \left[\frac{(m-1)^s + (-1)^s (p-1)}{p}\right] a_{r-s}(n),$$

where $a_{r-s}(n)$ is the number of operations in $\mathfrak{S}_n$ containing exactly $n - r + s$ cycles. Hence

$$G(m, p, n; t) = \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n-r+s}{s} m^{r-s} \left[\frac{(m-1)^s + (-1)^s (p-1)}{p}\right] a_{r-s}(n)\, t^r,$$

and so, upon putting $r = s + u$,

$$\begin{aligned}G(m, p, n; t) &= \sum_{u=0}^{n} \sum_{s=0}^{n-u} \binom{n-u}{s} m^u \left[\frac{(m-1)^s + (-1)^s (p-1)}{p}\right] a_u(n)\ t^{s+u}\\ &= \sum_{u=0}^{n} \left[\frac{\{1 + (m-1)t\}^{n-u} + (p-1)(1-t)^{n-u}}{p}\right] a_u(n)\, m^u t^u\\ &= \frac{1}{p}\left[\{1 + (m-1)t\} \sum_{u=0}^{n} a_u(n) \left(\frac{mt}{1 + (m-1)t}\right)^u \right.\\ &\qquad \left. + (p-1)(1-t)^n \sum_{u=0}^{n} a_u(n) \left(\frac{mt}{1-t}\right)^u\right]. \qquad (9.4)\end{aligned}$$

Now, from the recurrence relation

$$a_u(n+1) = a_u(n) + n\, a_{u-1}(n),$$

which may be verified by considering the partitions of $\mathfrak{S}_n$ which arise by sup-

pressing a fixed symbol in the permutations of $\mathfrak{S}_{n+1}$, it may be shown by induction that

$$\sum_{u=0}^{n} a_u(n)\,\xi^u = \prod_{r=1}^{n-1}(1+r\xi).$$

Hence, from 9.4,

$G(m, p, n; t)$

$$= \frac{1}{p}\Big[\{1+(m-1)t\}\prod_{r=1}^{n-1}\{1+(m-1)t+rmt\}+(p-1)(1-t)\prod_{r=1}^{n-1}(1-t+rmt)\Big]$$

$$= \frac{1}{p}\Big(\prod_{r=1}^{n-1}\{1+(rm-1)\,t\}\Big)\Big(\{1+(nm-1)\,t\}+(p-1)(1-t)\Big)$$

$$= \prod_{r=1}^{n-1}[1+(rm-1)\,t]\,.\,[1+(nq-1)\,t],$$

and so

$$G(m, p, n; t) = \prod_{i=1}^{n}(1+m_i t), \tag{9·5}$$

where the m_i are the exponents of the group in 9.3. Hence Theorem 5.3 is verified for these two groups.

(b) *The u.g.g.r. in U_2 (nos. (4) to (22) in Table VII).*

The verification of Theorem 5.3 for these groups is trivial, since in enumerating the groups in §4 we stated the number of reflections, and in each case this will be found to be equal to $m_1 + m_2$ where m_1 and m_2 have the values indicated in Table VII. Also, as the order of the group is $(m_1 + 1)(m_2 + 1)$, it follows that $g_2 = m_1 m_2$.

(c) *The u.g.g.r. in U_n (nos. (23) to (37) in Table VII).*

In order to determine the values of the g_r for these groups, we employ two different methods.

The first method depends upon examination of a polytope associated with the group and upon determining the number of operations of each type from the known geometry of the figure. This method can be applied easily only in a small number of dimensions ($n = 3, 4$) after which it is simpler to use the second method. In order to explain this it is necessary to make some definitions.

Let S be any operation of the group $\mathfrak{G}$ and suppose that it leaves invariant a linear space of dimension $n - r$ (and no space of higher dimension), then we shall call this linear space the *axis* of S. Thus the axis of a reflection is of $n - 1$ dimensions, and the axis of the identity consists of the whole space. All the operations of $\mathfrak{G}$ which leave a particular linear subspace L_{n-r} invariant (at least) form a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ which may be called the subgroup associated with that subspace. $\mathfrak{H}$ is a u.g.g.r. in r dimensions, and g_r (computed for $\mathfrak{H}$), which is the number of operations of $\mathfrak{H}$ which leave no more than a single point invariant, is evidently the number of operations of $\mathfrak{G}$ which leave *only* the L_{n-r}

invariant, i.e., have the L_{n-r} as their axis. We call this number the *multiplier* of $\mathfrak{H}$.

Turning our attention now to the collineation group $\mathfrak{G}'$ in S_{n-1} corresponding to $\mathfrak{G}$, we see that, for most of the groups with $n > 2$, the reflections of $\mathfrak{G}$ are in 1-1 correspondence with the homologies of $\mathfrak{G}'$ (§3). The reflections of $\mathfrak{G}$ that leave a space L_{n-r} invariant correspond to a set of homologies in $\mathfrak{G}'$ whose centres span a subspace S_{r-1} in S_{n-1}. Thus to each axis L_{n-r} corresponds a subspace of the configuration of centres of homologies of the collineation group, and since the latter have been enumerated for many of the larger groups **(11; 12; 13; 25)**, the determination of the former is straightforward.

By way of example, consider the group $[2\,1;\,3]^3$. The associated collineation group is of order 6,531,840 and we refer to Miss Hamill's description **(12)** of the configuration formed by the centres of the 126 homologies. The lines of the configuration are of two types: 2835 e-lines, each of which corresponds to an axis L_4 associated with the subgroup which consists of the direct product of two groups of order two, and 1680 κ-lines each of which corresponds to an axis L_4 associated with the dihedral group of order six. Considering the planes, spaces, etc. of the configuration we may identify and enumerate all the axes of $\mathfrak{G}$. Determination of the subgroup associated with each axis from the configuration of vertices in S_{r-1} is straightforward, but it is worth noting that the configurations β_r and γ_r of **(13)** correspond to axes associated with the groups $[3^{1,1,r-2}]$ and $[3^r]$ respectively, illustrating that β_2 and γ_2 are identical **(13, p. 58)**.

In Table VIII at the end of this paper we list all the axes of each type for the groups (23) to (37) by the method outlined above. The different subgroups $\mathfrak{H}$ associated with the axes are given in the second column of the table, and the multipliers in the third. The groups are denoted by the symbols of Table VII, and, as usual, the sign $\times$ implies that the direct product is to be taken. In addition, the symbol $[\Pi_n]$ is used for the symmetry group of the polytope Π_n.

10. The product of the generators. We now come to the verification of Theorem 5.4 which asserts that, in the case where $\mathfrak{G}$ is an irreducible u.g.g.r. in U_n generated by n reflections, these can be chosen in such a way that the period of their continued product is $h = m_n + 1$, and the characteristic roots of the corresponding matrix are $\exp(2\pi i m_r/h)$ $(r = 1, 2, \ldots, n)$, where m_i are the exponents of the group.

This has been established for the real groups by Coxeter **(10)**. It is trivial for the group generated by a single p-fold reflection in U_1, and in the case of the complex regular polygons in U_2 it may be verified that the pairs of generating reflections given in §4 have the required property.

The remaining groups to be considered are $G(m, m, n)$ and $G(m, 1, n)$, the groups of orders 336 and 2160 in U_3, the symmetry groups of the regular complex polytopes

$$3(24)3(24)3, \quad 3(24)3(18)2, \quad 3(24)3(24)3(24)3$$

and the groups $[2\,1;\,1]^4$, $[2\,1;\,2]^3$, $[2\,1;\,3]^3 \equiv [3\,1;\,2]^3$.

$G(m, 1, n)$ is the symmetry group of the regular polytope γ_n^m and may be generated **(24)** by the m-fold reflection

$$Q: \qquad x_1' = \theta x_1,\ x_i' = x_i \qquad (i = 2, 3, \ldots, n;\ \theta^{-1} = \exp(2\pi i/m))$$

and by the 2-fold reflections

$$R_i: \qquad x_i' = x_{i+1},\ x_{i+1}' = x_i \qquad (i = 1, 2, \ldots, n-1)$$

$$x_j' = x_j \qquad (j \neq i,\ i+1).$$

The product $QR_1R_2 \ldots R_{n-1}$ is the transformation

$$x_1' = \theta x_n,\ x_2' = x_1,\ x_3' = x_2,\ \ldots,\ x_n' = x_{n-1}.$$

The characteristic equation of the corresponding matrix is

$$\lambda^n - \theta = 0$$

and so the characteristic roots are the n values of $\theta^{1/n}$, or

$$\epsilon^{rm-1} \qquad (r = 1, 2, \ldots, n),$$

where $\epsilon = \exp(2\pi i/mn)$.

$G(m, m, n)$ is the symmetry of the polytope $\frac{1}{m}\gamma_n^m$ and may be generated **(24)** by the reflections R_i together with the 2-fold reflection

$$S: \qquad x_1' = \theta^{-1}x_2,\ \ x_2' = \theta x_1,\ \ x_i' = x_i \qquad (i = 3, 4, \ldots, n).$$

The product $SR_1R_2 \ldots R_{n-1}$ is the transformation

$$x'_1 = \theta^{-1}x_1,\ \ x_2' = \theta x_n,\ \ x_3' = x_2,\ \ x_4' = x_3,\ \ \ldots,\ \ x'_n = x_{n-1}.$$

The characteristic equation of the corresponding matrix is

$$(\lambda - \theta^{-1})(\lambda^{n-1} - \theta) = 0$$

and so the characteristic roots are θ^{-1} and the values of $\theta^{1/(n-1)}$, or

$$\epsilon^{n-1},\ \epsilon^{rm-1} \qquad (r = 1, 2, \ldots, n-1),$$

where $\epsilon = \exp(2\pi i/m(n-1))$.

The group of order 336 in U_3 **(19)** contains 2-fold reflections in 21 planes:

$$x_i = 0, \quad x_i \pm x_j = 0, \quad \alpha x_i \pm x_j \pm x_k = 0,$$

where (i, j, k) is any permutation of $(1, 2, 3)$ and α is either root of the equation $t^2 - t + 2 = 0$. We select

$$\alpha = \tfrac{1}{2}(1 - i\sqrt{7}) = -(\beta^4 + \beta^2 + \beta), \qquad \beta = \exp(2\pi i/7),$$

so that

$$\bar{\alpha} = \tfrac{1}{2}(1 + i\sqrt{7}) = -(\beta^3 + \beta^5 + \beta^6).$$

The group is generated by the reflections

$$10.1 \qquad \mathbf{R}_1: \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \quad \mathbf{R}_2: \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad \mathbf{R}_3: \tfrac{1}{2}\begin{pmatrix} 1 & -1 & -\alpha \\ -1 & 1 & -\alpha \\ -\bar{\alpha} & -\bar{\alpha} & 0 \end{pmatrix}$$

in the primes

$$x_3 - x_3 = 0, \quad x_3 = 0, \quad x_1 + x_2 + \alpha x_3 = 0,$$

respectively. The matrix of the product $R_1R_2R_3$ is

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & -\alpha \\ \bar{\alpha} & \bar{\alpha} & 0 \\ -1 & 1 & -\alpha \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^3 - \bar{\alpha}\lambda^2 - \alpha\lambda + 1 = 0,$$

and its characteristic roots are $\zeta^3, \zeta^5, \zeta^{13}$ where $\zeta = \exp(2\pi i/14)$.

The group of order 2160 in U_3 **(19)** contains 2-fold reflections in 45 planes:

$$x_i = 0, \quad x_i \pm \omega x_i = 0, \quad x_i \pm \gamma x_i \pm \gamma^2 x_k = 0,$$
$$x_i \pm \omega\gamma^2 x_j \pm \omega^2\gamma x_k = 0, \quad x_i \pm (1 - \omega^2\gamma)\, x_i \pm \omega x_k = 0,$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and γ is either root of the equation $t^2 + t - 1 = 0$. We substitute

$$\gamma = \tfrac{1}{2}(-1 + \sqrt{5}) = 2\cos(2\pi/5) = \tau^{-1},$$

where $\tau = \frac{1}{2}(1 + \sqrt{5}) = -2\cos(4\pi/5)$. The group is generated by the reflections

10.2 $$\mathbf{R}_1: \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \mathbf{R}_2: \frac{1}{2}\begin{pmatrix} 1 & -\omega\tau & -\omega^2\tau^{-1} \\ -\omega^2\tau & -\tau^{-1} & -\omega \\ -\omega\tau^{-1} & -\omega^2 & \tau \end{pmatrix}, \quad \mathbf{R}_3: \begin{pmatrix} & -\omega^2 & \\ -\omega & & \\ & & 1 \end{pmatrix}$$

in the primes $x_1 = 0$, $\omega^2\tau x_1 + \tau^2 x_2 + \omega x_3 = 0$, $x_1 + \omega^2 x_2 = 0$ respectively. The matrix of the product $R_1R_2R_3$ is

$$\frac{1}{2}\begin{pmatrix} -\omega^2\tau & \omega^2 & \omega^2\tau^{-1} \\ \omega\tau^{-1} & \omega\tau & -\omega \\ 1 & \tau^{-1} & \tau \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^3 + \omega^2\tau\,\lambda^2 + \omega\tau\,\lambda + 1 = 0$$

and its characteristic roots are $\zeta^5, \zeta^{11}, \zeta^{29}$, where $\zeta = \exp(2\pi i/30)$.

The symmetry group of the polyhedron $3(24)3(24)3$ **(23)** contains 3-fold reflections in the 12 planes:

$$x_i = 0, \quad x_1 + \omega^j x_2 + \omega^k x_3 = 0 \qquad (i, j, k = 1, 2, 3).$$

It is generated by the reflections

10.3 $$\mathbf{R}_1: \begin{pmatrix} 1 & & \\ & 1 & \\ & & \omega^2 \end{pmatrix}, \quad \mathbf{R}_2: \frac{-i}{\sqrt{3}}\begin{pmatrix} \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{pmatrix}, \quad \mathbf{R}_3: \begin{pmatrix} 1 & & \\ & \omega^2 & \\ & & 1 \end{pmatrix}$$

in the planes $x_3 = 0$, $x_1 + x_2 + x_3 = 0$, $x_2 = 0$ respectively. The characteristic

roots of $\mathbf{R}_1$ and $\mathbf{R}_3$ are $(1, 1, \omega^2)$ and of $\mathbf{R}_2$ are $(1, 1, \omega)$. The matrix of the product $R_1 R_2^{-1} R_3$ is

$$\frac{i}{\sqrt{3}}\begin{pmatrix} \omega^2 & 1 & \omega \\ \omega & \omega & \omega \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^3 - \omega^2\lambda^2 + \omega\lambda - 1 = 0,$$

and its characteristic roots are $\zeta^5, \zeta^8, \zeta^{11}$ where $\zeta = \exp(2\pi i/12)$.

The symmetry group **(23)** of the regular polyhedron $3(24)3(18)2$ (or its reciprocal) contains in addition to the above 3-fold reflections, 2-fold reflections in the nine planes

$$x_i - \omega^k x_j = 0, \qquad (i, j, k = 1, 2, 3).$$

It is generated by the 3-fold reflections R_1, R_2 above together with the 2-fold reflection

10.4 $$\mathbf{S}: \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

in the plane $x_2 - x_3 = 0$. The matrix of the product $SR_1R_2^{-1}$ is

$$\frac{i}{\sqrt{3}}\begin{pmatrix} \omega^2 & \omega & \omega \\ 1 & 1 & \omega \\ \omega & \omega^2 & \omega \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^2 + \omega = 0,$$

and its characteristic roots are $\zeta^5, \zeta^{11}, \zeta^{17}$ where $\zeta = \exp(2\pi i/18)$.

The symmetry group of the regular polytope $3(24)3(24)3(24)3$ **(8; 23)** contains 80 3-fold reflections in 40 primes:

$$x_i = 0, \quad x_1 + \omega^j x_2 + \omega^k x_3 = 0, \quad x_1 - \omega^j x_2 - \omega^k x_4 = 0$$
$$x_1 - \omega^j x_3 + \omega^k x_4 = 0, \quad x_2 - \omega^j x_3 - \omega^k x_4 = 0$$
$$(i = 1, 2, 3, 4; \quad j, k = 1, 2, 3).$$

The group is generated by the 3-fold reflections:

10.5
$$\mathbf{R}_1: \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \omega^2 & \\ & & & 1 \end{pmatrix}, \quad \mathbf{R}_2: \frac{-i}{\sqrt{3}}\begin{pmatrix} \omega & \omega^2 & \omega^2 & 0 \\ \omega^2 & \omega & \omega^2 & 0 \\ \omega^2 & \omega^2 & \omega & 0 \\ 0 & 0 & 0 & i\sqrt{3} \end{pmatrix}$$
$$\mathbf{R}_3: \begin{pmatrix} 1 & & & \\ & \omega^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \mathbf{R}_4: \frac{-i}{\sqrt{3}}\begin{pmatrix} \omega & -\omega^2 & 0 & -\omega^2 \\ -\omega^2 & \omega & 0 & \omega^2 \\ 0 & 0 & i\sqrt{3} & 0 \\ -\omega^2 & \omega^2 & 0 & \omega \end{pmatrix}$$

in the primes $x_3 = 0$, $x_1 + x_2 + x_3 = 0$, $x_2 = 0$, $-x_1 + x_2 + x_4 = 0$ respectively. The characteristic roots of $\mathbf{R}_1$, $\mathbf{R}_3$ are $(1, 1, 1, \omega^2)$ and of $\mathbf{R}_2$, $\mathbf{R}_4$ are $(1, 1, 1, \omega)$. The matrix of the product $R_1R_3^{-1}R_2R_4^{-1}$ is

$$\frac{-i}{\sqrt{3}}\begin{pmatrix} 0 & \omega & -\omega & -\omega^2 \\ -\omega & -\omega & -\omega & 0 \\ \omega & 0 & -\omega & \omega^2 \\ \omega & -\omega & 0 & -\omega^2 \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^4 - \omega^2\lambda^3 + \omega\lambda^2 - \lambda + \omega^2 = 0,$$

and its characteristic roots are ζ^{11}, ζ^{17}, ζ^{23}, ζ^{29}, where $\zeta = \exp(2\pi i/30)$.

The group $[2\,1;\,1]^4$ can be generated **(24, 373)** by 2-fold reflections P_2, P_1, Q_1, R_1 in the primes $x_1 + x_2 + x_3 + x_4 = 0$, $x_1 - ix_2 = 0$, $x_1 - x_2 = 0$, $x_2 - x_3 = 0$. The matrix of the product $P_2P_1Q_1R_1$ is

$$\frac{1}{2}\begin{pmatrix} i & -1 & i & -1 \\ -i & -1 & -i & -1 \\ -i & 1 & i & -1 \\ -i & -1 & i & 1 \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^4 - i\lambda^3 - \lambda^2 + i\lambda + 1 = 0,$$

and its characteristic roots are ζ^3, ζ^7, ζ^{11}, ζ^{19} where $\zeta = \exp(2\pi i/20)$.

The group $[2\,1;\,2]^3$ can be generated **(24, 373)** by 2-fold reflections P_1, P_2, Q_1, R_1, R_2 in the primes

$$x_2 - x_3 = 0, \quad x_3 - x_4 = 0, \quad x_1 - x_2 = 0, \quad x_1 - \omega x_2 = 0,$$
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$$

(using coordinates in six dimensions). The matrix of the product $R_2R_1Q_1P_1P_2$ is

$$\frac{1}{3}\begin{pmatrix} 2\omega & -1 & -1 & -\omega^2 & -1 & -1 \\ -\omega & -1 & -1 & 2\omega^2 & -1 & -1 \\ -\omega & 2 & -1 & -\omega^2 & -1 & -1 \\ -\omega & -1 & 2 & -\omega^2 & -1 & -1 \\ -\omega & -1 & -1 & -\omega^2 & 2 & -1 \\ -\omega & -1 & -1 & -\omega^2 & -1 & 2 \end{pmatrix}.$$

Its characteristic equation, after division by the extraneous factor $1 - \lambda$ arising from the extra dimension, is

$$\lambda^5 - \omega\lambda^4 + \omega^2\lambda^3 + \omega\lambda^2 - \omega^2\lambda + 1 = 0,$$

and its characteristic roots are ζ^3, ζ^5, ζ^9, ζ^{11}, ζ^{17} where $\zeta = \exp(2\pi i/18)$.

The group $[3\,1;\,2]^3$ can be generated **(24, 373)** by the above five reflections together with the 2-fold reflection P_3 in the prime $x_4 - x_5 = 0$. The matrix of the product $R_2R_1Q_1P_1P_2P_3$ is

$$\frac{1}{3}\begin{pmatrix} 2\omega & -1 & -1 & -1 & -\omega^2 & -1 \\ -\omega & -1 & -1 & -1 & 2\omega^2 & -1 \\ -\omega & 2 & -1 & -1 & -\omega^2 & -1 \\ -\omega & -1 & 2 & -1 & -\omega^2 & -1 \\ -\omega & -1 & -1 & 2 & -\omega^2 & -1 \\ -\omega & -1 & -1 & -1 & -\omega^2 & 2 \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^6 - \omega\lambda^5 + \omega^2\lambda^4 - \lambda^3 + \omega\lambda^2 - \omega^2\lambda + 1,$$

and its characteristic roots are $\zeta^5, \zeta^{11}, \zeta^{17}, \zeta^{23}, \zeta^{29}, \zeta^{41}$ where $\zeta = \exp(2\pi i/42)$.

This completes the verification of Theorem 5.4 for all the irreducible u.g.g.r. generated by n reflections.

11. Abstract definitions of the finite unitary groups generated by n reflections. We now give, either explicitly or by reference, abstract definitions of all the u.g.g.r. which are generated by n reflections. In each case the abstract definition takes the form of n generators (corresponding to the n generating reflections of the unitary group) and a set of relations that they satisfy. (In certain cases it will be convenient to introduce further elements of the group into the definition in order to simplify it. Such elements will always be denoted by Z_i.)

The definitions of the real groups ($G(2, 2, n)$, $G(2, 1, n)$ and nos. (1), (23), (28), (30), (35), (36), (37) of Table VII) have been given by Coxeter **(8; 9)** who shows how they may be read off from the graphical symbol for the spherical simplex that forms a fundamental region for the group.

For the symmetry groups of the regular polygons (nos. (4), (5), (6), (8), (9), (10), (14), (16), (17), (18), (20), (21)) the definitions are implicit in the tables of §4.

Shephard **(24)** has given definitions for $G(m, m, n)$, $G(m, 1, n)$ and nos. (29), (33), (34) by extending Coxeter's graphical notation to these unitary groups and reading off the definitions in an analogous manner.

The abstract definitions of the remaining groups are as follows:

(24) The operations R_1, R_2, R_3 of 10.1 satisfy the relations

$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^4 = (R_2R_3)^4 = (R_3R_1)^3 = (R_1R_2R_1R_3)^3 = 1,$$

and this is an abstract definition for the u.g.g.r. of order 336 in U_3. The subgroup $\{R_3R_1, R_1R_2\}$ is the simple G_{168}, since it is $(3, 4 \mid 4, 3) \sim (3, 3 \mid 4, 4)$ in the notation of **(7**, pp. 78, 83**)**.

(27) The operations R_1, R_2, R_3 of 10.2 satisfy the relations

$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_1)^4 = (R_1R_2R_1R_3)^5 = 1,$$

and this is an abstract definition for the u.g.g.r. of order 2160 in U_3. The subgroup $\{R_1R_2, R_2R_3\}$ is $(3, 3 \mid 4, 5)$ in the notation of **(7**, p. 85**)**.

(25) The operations R_1, R_2, R_3 of 10.3 satisfy the relations

$$R_1^3 = R_2^3 = R_3^3 = 1, \quad R_1R_3 = R_3R_1,$$
$$(R_1R_2)^2 = Z_1, \quad Z_1R_1 = R_1Z_1, \quad Z_1^2 = 1,$$
$$(R_2R_3)^2 = Z_2, \quad Z_2R_2 = R_2Z_2, \quad Z_2^2 = 1,$$

and this is an abstract definition for the symmetry group (of order 648) of the regular polyhedron $3(24)3(24)3$.

(26) The operations S, R_1, R_2 of 10.3 and 10.4 satisfy the relations

$$S^2 = R_1^3 = R_2^3 = 1, \quad SR_2 = R_2S,$$
$$(SR_1)^2 = Z_1, \quad Z_1R_1 = R_1Z_1, \quad Z_1^3 = 1,$$
$$(R_1R_2)^2 = Z_2, \quad Z_2R_2 = R_2Z_2, \quad Z_2^2 = 1,$$

and this is an abstract definition for the symmetry group (of order 1296) of the regular polyhedron $3(24)3(18)2$ or its reciprocal.

(32) The operations R_1, R_2, R_3, R_4 of 10.5 satisfy the relations

$$R_1^3 = R_2^3 = R_3^3 = R_4^3 = 1, \quad R_1R_3 = R_3R_1, \quad R_1R_4 = R_4R_1, \quad R_2R_4 = R_4R_2,$$
$$(R_1R_2)^2 = Z_1, \quad Z_1R_1 = R_1Z_1, \quad Z_1^2 = 1,$$
$$(R_2R_3)^2 = Z_2, \quad Z_2R_2 = R_2Z_2, \quad Z_2^2 = 1,$$
$$(R_3R_4)^2 = Z_3, \quad Z_3R_3 = R_3Z_3, \quad Z_3^2 = 1,$$

and this is an abstract definition for the symmetry group (of order 155520) of the regular polytope $3(24)3(24)3(24)3$.

Each of the above definitions has been checked by the Todd-Coxeter method of enumeration by cosets **(27)**. It is worth noting that the definitions of the groups (25), (26) and (32) are what might have been expected by applying the rules of **(24**, p. 374**)** and the definitions of the groups $G(3, 3, 2)$ and (4) to the extended Coxeter graph **(24**, p. 368**)**.

The authors wish to thank Professor Coxeter for removing some redundant relations. He observes that the definition of (26) still contains one such relation: $Z_1^3 = 1$. In fact, the remaining relations imply

$$Z_1^3 = Z_1^3 R_1^{-3} = (Z_1R_1^{-1})^3 = (SR_1S)^3 = SR_1^3S = 1.$$

Thus a sufficient set of generating relations for this G_{1296} is

$$S^2 = R_1^3 = R_2^3 = 1, \quad SR_2 = R_2S,$$
$$(SR_1)^2 = (R_1S)^2,$$
$$(R_1R_2)^2 = (R_2R_1)^2 = Z, \quad Z^2 = 1.$$

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TABLE VII

LIST OF IRREDUCIBLE U.G.G.R. IN U_n

No.	n	Symbol	Polytope	g	g'	$m_i + 1$	Reference
1	n	$[3^{n-1}]$	α_n	$(n+1)!$	$(n+1)!$	$2, 3, \ldots, n+1$	(10)
2	n	$G(m, p, n)$		$qm^{n-1}n!$	$m^{n-1}n!/d^n$	$m, 2m, \ldots, (n-1)m, qn$	§§2, 6
3	1	$[\ ]^m$	m-line	m	1	m	§§3, 6
4	2		$3(24)3$	24	12	4, 6	§4
5	2		$3(72)3$	72	12	6, 12	"
6	2		$3(48)2$	48	12	4, 12	"
7	2			144	12	12, 12	"
8	2		$4(96)4$	96	24	8, 12	"
9	2		$4(192)2$	192	24	8, 24	"
10	2		$4(288)3$	288	24	12, 24	"
11	2			576	24	24, 24	"
12	2			48	24	6, 8	"
13	2			96	24	8, 12	"
14	2		$3(144)2$	144	24	6, 24	"
15	2			288	24	12, 24	"
16	2		$5(600)5$	600	60	20, 30	"
17	2		$5(1200)2$	1200	60	20, 60	"
18	2		$5(1800)3$	1800	60	30, 60	"
19	2			3600	60	60, 60	"
20	2		$3(360)3$	360	60	12, 30	"
21	2		$3(720)2$	720	60	12, 60	"
22	2			240	60	12, 20	"
23	3	[3, 5]	[3, 5]	120	60	2, 6, 10	(10)
24	3			336	168	4, 6, 14	§§3, 6
25	3		$3(24)3(24)3$	648	216	6, 9, 12	"
26	3		$3(24)3(18)2$	1296	216	6, 12, 18	"
27	3			2160	360	6, 12, 30	"
28	4	[3, 4, 3]	[3, 4, 3]	1152	576	2, 6, 8, 12	(10)
29	4	$[2\ 1;1]^3$	$(\frac{1}{3}\gamma_4^2)^{+1}$	7680	1920	4, 8, 12, 20	§§3, 6
30	4	[3, 3, 5]	[3, 3, 5]	14400	7200	2, 12, 20, 30	(10)
31	4		$(\frac{1}{3}\gamma_4^3)^{+1}$	64.6!	11520	8, 12, 20, 24	§§3, 6
32	4		$3(24)3(24)3(24)3$	216.6!	36.6!	12, 18, 24, 30	"
33	5	$[2\ 1;2]^3$	$(\frac{1}{3}\gamma_4^2)^{+1}$	72.6!	36.6!	4, 6, 10, 12, 18	"
34	6	$[2\ 1;3]^3$	$(\frac{1}{3}\gamma_4^3)^{+1}$	103.9!	18.9!	6, 12, 18, 24, 30, 42	"
35	6	$[3^{2,1,1}]$	2_{21}	72.6!	72.6!	2, 5, 6, 8, 9, 12	(10)
36	7	$[3^{2,2,1}]$	3_{21}	8.9!	4.9!	2, 6, 8, 10, 12, 14, 18	"
37	8	$[3^{3,2,1}]$	4_{21}	192.10!	96.10!	2, 8, 12, 14, 18, 20, 24, 30	"

* $m = pq$, $m > 1$, $n > 1$, $d = (p, n)$.

The u.g.g.r. in U_n are all generated by n reflections except for the following: $G(m, p, n)$ for $p \neq 1, m$; the groups in U_2 (nos. 7, 11, 12, 13, 15, 19, 22) which are not the symmetry groups of regular polygons; and the group of the polytope $(\frac{1}{3}\gamma_4^3)^{+1}$ in U_6.

TABLE VIII

TYPES OF OPERATION IN U.G.G.R.

Dimension of axis	Subgroup Φ	Multiplier of Φ	Group (23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)	(37)
		$g_n =$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$n-1$	$[\ ]$	1	15	21		9	45	24	40	60	60		45	126	36	63	120
	$[\ ]^2$	2			12	12						40					
		$g_{n-1} =$	15	21	24	33	45	24	40	60	60	80	45	126	36	63	120
$n-2$	$[\ ]\times[\ ]$	1	15					72	120	450	360		270	2,835	270	945	3,780
	[3]	2	10	28			120	32	160	200	320		240	1,680	120	336	1,120
	[4]	3		21			45	18	30								
	[5]	4	6				36			72							
	$[\ ]^2\times[\ ]$	2				36											
	$[3(24)3]$	15			9	9						90					
	$[\ ]^2\times[\ ]^2$	4			12							240					
	$G(3,1,2)$	10				12											
	$G(4,2,2)$	9									30						
		$g_{n-1} =$	59	119	183	327	519	190	530	1,138	1,270	2,310	750	6,195	510	1,617	6,020
$n-3$	$[\ ]\times[\ ]\times[\ ]$	1											270	11,340	540	4,095	37,800
	$[3]\times[\ ]$	2						96	160	600	960		720	30,240	720	5,040	40,320
	$[5]\times[\ ]$	4								360							
	$[3^2]$	6							160	300	480		540	11,340	270	1,260	7,560
	[4, 3]	15						24	40								
	[5, 3]	45								60							
	$[3(24)3]\times[\ ]^2$	30										360					
	$[3(24)3(24)3]$	440										40					
	$G(3,3,3)$	20											40	560			
	$G(4,4,3)$	42							20								
	$G(4,2,3)$	105									60						
		$g_{n-1} =$						552	2,720	7,140	11,100	28,400	5,750	151,080	3,000	21,735	163,800
$n-4$	$[\ ]\times[\ ]\times[\ ]\times[\ ]$	1														3,780	113,400
	$[3]\times[\ ]\times[\ ]$	2												45,360	1,080	15,120	302,400
	$[3]\times[3]$	4												30,240	120	3,360	67,200
	$[3^2]\times[\ ]$	6											540	66,040	540	8,820	151,200
	$[3^3]$	24											216	27,216	216	2,016	24,192
	$G(2,2,4)$	45											45	2,835	45	315	3,150
	$G(3,3,3)\times[\ ]$	20												5,040			
	$G(3,3,4)$	240											40	1,680			
		$g_{n-4} =$											20,049	1,904,679	13,089	162,939	2,616,558

	g_{n-4} =												20,049	1,904,679	13,089	162,939	2,616,558

Dimension of axis	Subgroup Φ	Multiplier of Φ	Group (23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)	(37)
$n-5$	$[3]\times[\,]\times[\,]\times[\,]$	2														5,040	604,800
	$[3]\times[3]\times[\,]$	4													360	10,080	403,200
	$[3^2]\times[\,]\times[\,]$	6														7,560	453,600
	$[3^2]\times[3]$	12												45,360		5,040	302,400
	$[3^3]\times[\,]$	24												27,216	216	6,048	241,920
	$[3^4]$	120												18,144	36	1,344	40,320
	$G(2,2,4)\times[\,]$	45														945	37,800
	$G(2,2,5)$	420												3,402	27	378	7,560
	$G(3,3,4)\times[\,]$	240												5,040			
	$G(3,3,5)$	3,520												672			
	$[2\,1;2]^3$	25,245												120			
	g_{n-5} =													11,559,534	22,284	663,957	24,093,480
$n-6$	$[3]\times[3]\times[\,]\times[\,]$	4															604,800
	$[3^2]\times[3]\times[\,]$	12														5,040	604,800
	$[3^3]\times[\,]\times[\,]$	24															362,880
	$[3^3]\times[3^2]$	36															151,200
	$[3^3]\times[3]$	48														2,016	241,920
	$[3^4]\times[\,]$	120														1,008	120,960
	$[3^5]$	720														288	34,560
	$G(2,2,3)\times[3]$	90															50,400
	$G(2,2,5)\times[\,]$	420														378	45,360
	$G(2,2,6)$	4,725														63	3,780
	$[3^{1,\,1,\,2}]$	12,320														28	1,120
	g_{n-6} =															1,289,963	130,085,780
$n-7$	$[3^3]\times[3]\times[\,]$	48															241,920
	$[3^4]\times[3^2]$	144															120,960
	$[3^4]\times[\,]$	720															34,560
	$[3^6]$	5,040															8,640
	$G(2,2,5)\times[3]$	840															30,240
	$G(2,2,7)$	82,370															1,080
	$[3^{1,\,1,\,2}]\times[\,]$	12,320															3,360
	$[3^{1,\,1,\,3}]$	765,765															120
	g_{n-7} =																323,507,400
	g_0 =		45	195	440	935	1,595	385	4,389	6,061	33,649	124,729	25,245	25,569,445	12,320	765,765	215,656,441
Order of group = Σg_i =			120	336	648	1,296	2,160	1,152	7,680	14,400	46,080	155,520	51,840	39,191,040	51,840	2,903,040	696,729,600

References

1. G. Bagnera, *I gruppi finiti di trasformazioni lineari dello spazio che contengono omologie*, Rend. Circ. Mat. Palermo, *19* (1905), 1–56.
2. H. F. Baker, *A locus with* 25920 *linear self transformations* (Cambridge, 1946).
3. H. F. Blichfeldt, *The finite discontinuous primitive groups of collineations in four variables*, Math. Annalen, *60* (1905), 204–231.
4. H. Burkhardt, *Untersuchungen auf dem Gebiete der hyperelliptischen Modulfunctionen (Zweiter Teil)*, Math. Annalen, *38* (1891), 161–224.
5. W. Burnside, *Theory of groups of finite order* (second edition, Cambridge, 1911).
6. H. S. M. Coxeter, *Discrete groups generated by reflections*, Ann. Math., *35* (1934), 588–621.
7. ———, *The abstract groups* $G^{m,n,p}$, Trans. Amer. Math. Soc. *45* (1939), 73–150.
8. ———, *The polytope* 2_{21} *whose twenty seven vertices correspond to the lines on the general cubic surface*, Amer. J. Math., *62* (1940), 457–486.
9. ———, *Regular polytopes* (London, 1948; New York, 1949).
10. ———, *The product of the generators of a finite group generated by reflections*, Duke Math. J., *18* (1951), 765–782.

10a. J. S. Frame, *The classes and representations of the groups of* 27 *lines and* 28 *bitangents*, Annali di Math., *32* (1951), 83–119.

11. C. M. Hamill, *The finite primitive collineation groups which contain homologies of period two* (Thesis, University of Cambridge, 1950).
12. ———, *On a finite group of order* 6,531,840, Proc. London Math. Soc. (2), *52* (1951), 401–454.
13. ———, *A collineation group of order* $2^{13}.3^5.5^2.7$, Proc. London Math. Soc. (3), *3* (1953), 54–79.
14. F. Klein, *Ueber die Transformationen siebenter Ordnung der elliptischen Funktionen*, Math. Annalen, *14* (1879), 428–471.
15. ———, *Lectures on the icosahedron* (trans. Morrice), (London, 1913).
16. H. Maschke, *Ueber die quaternäre, endliche, lineare Substitutionsgruppe der Borchardt'schen Moduln*, Math. Annalen, *30* (1887), 496–515.
17. ———, *Aufstellung des vollen Formensystems einer quaternären Gruppe von* 51840 *linearen substitutionen*, Math. Annalen, *33* (1889), 317–344.
18. G. A. Miller, H. F. Blichfeldt and L. E. Dickson, *Theory and applications of finite groups* (New York, 1916).
19. H. H. Mitchell, *Determination of the ordinary and modular ternary linear groups*, Trans. Amer. Math. Soc., *12* (1911), 207–242.
20. ———, *Determination of the finite quaternary linear groups*, Trans. Amer. Math. Soc., *14* (1913), 123–142.
21. ———, *Determination of all primitive collineation groups in more than four variables which contain homologies*, Amer. J. of Math., *36* (1914), 1–12.
22. T. Molien, *Ueber die Invarianten der linearen Substitutionsgruppe*, Berliner Sitzungsber., (1898), 1152–1156.
23. G. C. Shephard, *Regular complex polytopes*, Proc. London Math. Soc. (3), *2* (1952), 82–97.
24. ———, *Unitary groups generated by reflections*, Can. J. Math., *5* (1953), 364–383.
25. J. A. Todd, *On the simple group of order* 25920, Proc. Royal Soc. (A), *189* (1947), 326–358.
26. ———, *The invariants of a finite collineation group in five dimensions*, Proc. Cambridge Phil. Soc., *46* (1950), 73–90.
27. J. A. Todd and H. S. M. Coxeter, *A practical method for enumerating cosets of a finite abstract group*, Proc. Edinburgh Math. Soc. (2), *5* (1936), 26–34.
28. H. Valentiner, *De endelige Transformations-Gruppers Theori*, K. danske vidensk, selsk. (Copenhagen) (6), *5* (1889), 64–235.
29. A. Wiman, *Ueber eine einfache Gruppe von* 360 *ebenen Collineationen*, Math. Annalen, *47* (1896), 531–556.

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